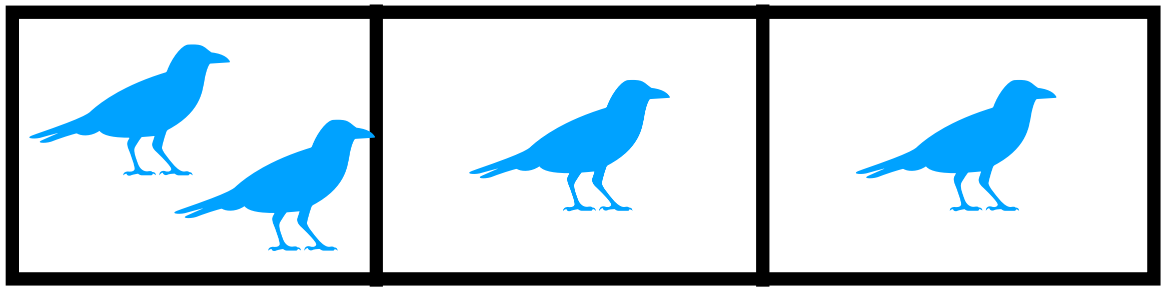
Methods of Counting

6.1-6.6

The Pigeonhole Principle

Example: **A 3-box Pigeon coop and 4 pigeons**



Definition: *Pigeonhole Principle*

If n items are placed in k boxes, at least one box contains at least $\lceil \frac{n}{k} \rceil$ items

Definition: *Pigeonhole Principle (w/functions)*

Let $f : X \to Y$, $|X| = n$, $|Y| = k$, and $m = \lceil \frac{n}{k} \rceil$.

There are at least m values such that

$f(a_1) = f(a_2) = \ldots = f(a_m)$

The Pigeonhole Principle

Example:

The last week of the semester has just 3 days of class meetings but you have 7 assignments due that week. By the pigeonhole principle, at least on day has at least $\lceil \frac{7}{3} \rceil$ = 3 assignments due.

How many contacts must be in your cell phone to ensure that 2 last names begin with the same pair of letters?

Answer: $26^2 + 1 = 676 + 1 = 677$

The Multiplication Principle

Example:

How many possible 3-digit octal numbers are there?

Answer: 8 8 8 $\Rightarrow 8 \cdot 8 \cdot 8 = 8^3 = 2^{3^3} = 2^9 = 512$

Definition: *Multiplication Principle (a.k.a. Product Rule)*

If there are s steps in an activity, with n_x ways to accomplish step x, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_s$ ways to complete all s steps.

For the Octal example, $s = 3$ and $n_1 = n_2 = n_3 = 8$

The Multiplication Principle

Example:

Party choices: 3 to choose from on Thrusday, 6 on Friay, 5 on Saturday, and 2 on Sunday. If you attend only one party per night, how many party schedules can be created?

Answer: By the M.P. $3 \cdot 6 \cdot 5 \cdot 2 = 180$ schedules

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Now consider three digital octal numbers without digit reuse. How many such values are there?

Answer: By the M.P. $8 \cdot 7 \cdot 6 = 336$

Note: $|P_1 \times P_2 \times \ldots \times P_s| = |P_1| \cdot |P_2| \cdot \ldots \cdot |P_s|$.

The Addition Principle

Definition: *Addition Principle (a.k.a. Product Rule)*

> If there are t tasks, with n_x ways to accomplish the x^{th} task, there are $n_1 + n_2 + \ldots + n_t$ ways to accomplish one of these tasks, assuming that the tasks are non-interfering.

Example:

You need to enroll in a literature class. 4 English Lit, 3 Poetry, and 5 World Lit classes fit your schedule.

By the A.P, there are $4 + 3 + 5 = 12$ possible ways for you to enroll in a Lit class.

The Addition Principle

Example:

Grade Sheet Identifiers 4-8 printable ASCII characters

How many IDs of 4 letters are there?

How many ID's of 5 letters are there?

But you can choose one of any of the 5 legal lengths.

By the A.P., there are

$$\sum_{i=4}^{8} 95^i = 6{,}704{,}780{,}953{,}650{,}625$$

(6.7 quadrillion!) possible grade sheet identifiers

The Principle of Inclusion-Exclusion

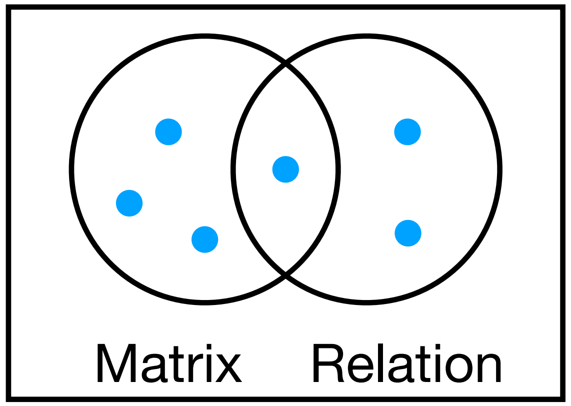
- A problem with the Addition Principle:
  - "Non-interfering" - no overlapping of tasks may occur!

Example:

I need a quiz question. I have four questions about matrices and three about relations. But if one is about matrix representation of relations, it is a member of both groups.

$\Rightarrow$ The Addition Principle does not apply!

(It reports $4 + 3 = 7$, but there are only 6 questions - the intersecting question is being counted twice.)



The Principle of Inclusion-Exclusion

Definition: *Principle of Inclusion-Exclusion for Two Sets*

The cardinality of the union of sets M and N is the sum of their individual cardinalities, excluding the cardinality of their intersection

That is: $|M \cup N| = |M| + |N| = |M \cap N|$

$$
\begin{aligned}
|\textbf{Matrix} \cup \textbf{Relation}| &= |\textbf{Matrix}| + |\textbf{Relation}| - |\textbf{Matrix} \cap \textbf{Relation}| \\
&= 4 + 3 - 1 \\
&= 6
\end{aligned}
$$

The Principle of Inclusion-Exclusion

Definition: *Principle of Inclusion-Exclusion for Three Sets*

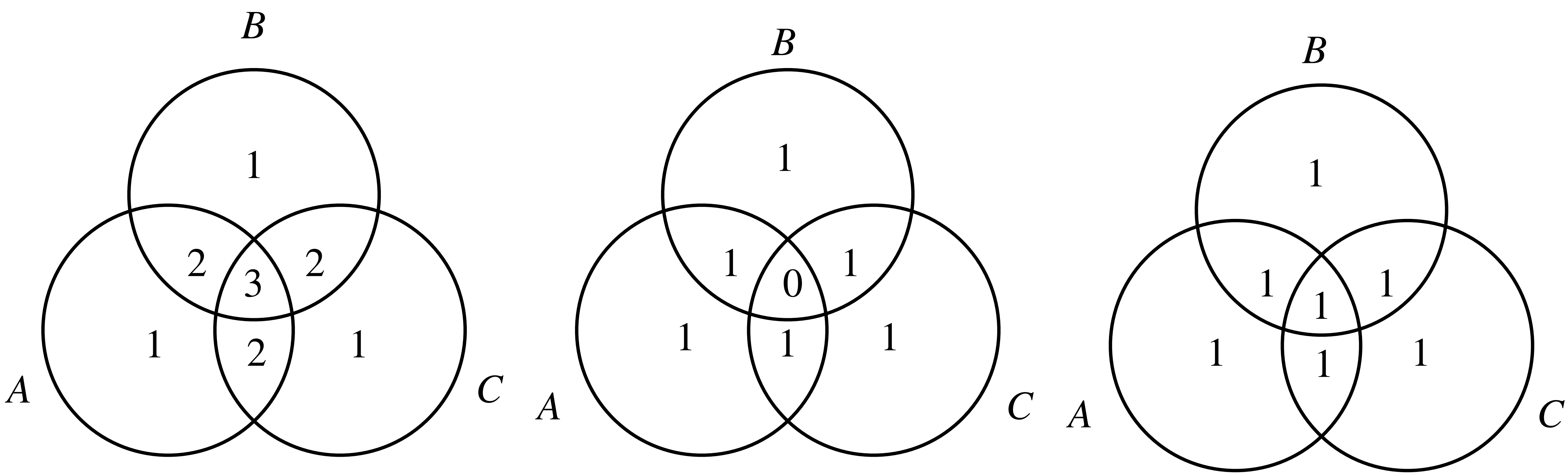
The cardinality of the union of sets M, N, and O is the sum of their individual cardinalities, <u>excluding</u> the sum of the cardinalities of their pairwise intersections but <u>including</u> the cardinality of their intersection

That is:
$$\begin{aligned}|M \cup N \cup O| &= |M| + |N| + |O| \\ &\quad - |M \cap N| - |M \cap O| - |N \cap O| \\ &\quad + |M \cap N \cap O|\end{aligned}$$

And, of course, this can be extended beyond 3 sets.

The Principle of Inclusion-Exclusion

- Why so complex?



$$|A| + |B| + |C| \quad - |A \cap B| - |A \cap C| - |B \cap C| \quad + |A \cap B \cap C|$$

The Principle of Inclusion-Exclusion

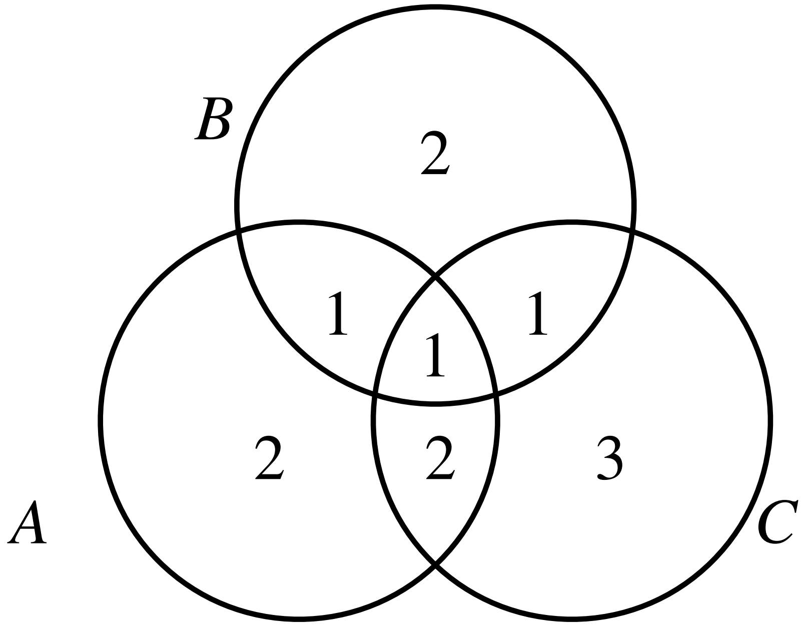
Example:

Let $|A| = 6$, $|B| = 5$, $|C| = 7$, $|A \cap B| = 2$, $|A \cap C| = 3$, $|B \cap C| = 2$ and $|A \cap B \cap C| = 1$.

What is $|A \cup B \cup C|$?

$|A \cup B \cup C| = 6 + 5 + 7 - (2 + 3 + 2) + 1 = 12$ items

Hint: Fill the Venn diagram from the center and work outward.



Permutations

Definition: *Permutation*

An ordering of $n \geq 0$ distinct elements.

Example:

Consider a golf tournament with a 5-way playoff between players A, B, C, D, and E. To determine the order of play they draw #s from a hat.

This generates a permutation of the players …

… but how many possible permutations are there?

Permutations

Conjecture: There are $n!$ possible permutations of n elements

Proof (direct):

There are n ways to select the 1st element.

$n - 1$ ways to select the 2nd, etc.

By the multiplication principle, the number of possible orderings is $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$

Therefore, there are $n!$ possible permutations of n elements.

r-Permutations

Definition: *r-Permutation* $P(n, r)$

An ordering of an r-element subset of n distinct elements is called an r-permutation.

Conjecture: **The number of r- permutations of n elements denoted $P(n, r)$, is $n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$, $r \leq n$**

Proof Outline:

1st 2nd r-th

$$n \cdot (n-1) \cdot \ldots \cdot (n-(r-1))$$

$$\ldots \cdot (n-r+1)$$

r-Permutations

Observation:

$$n \cdot (n-1) \cdot \ldots \cdot (n-r+1) \cdot \boxed{(n-r) \cdot \ldots \cdot 2 \cdot 1} = n!$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example:

How many 3-permutations can be formed from 5 elements?

$n - r + 1 = 5 - 3 + 1 = 3$ and $5 * 4 * 3 = 60$

Or: $P(5,3) = \dfrac{n!}{n-r}! = \dfrac{5!}{2!} = \dfrac{5 * 4 * 3 * 2 * 1}{2 * 1} = 60$

r-Permutations

Example:

16 countries are competing for medals (gold, silver, and bronze) in Team Discrete math at the Olympics. In how many was can medals be awarded?

Answer: $P(16,3) = \dfrac{16!}{13!} = 16 \cdot 15 \cdot 14 = 3360$

r- Combination

Definition: *r-Combination*

An r-Combination of an n-element set X is an r-element subset of X. The quantity of r- element subsets is denoted $C(n, r)$ or $\binom{n}{r}$, and is read "n choose r

Other Notations: ${}_nC_r$ $\quad C_{r,n}$

Example:

In how many ways my 2-element subsets be chosen from $\{A, B, C\}$?

Answer: Order does not matter in sets, so $\binom{3}{2} = 3$

The sets: $\{A, B\}$, $\{A, C\}$, and $\{B, C\}$.

Note that $P(3,2) = 6$.

r- Combination

The r-Permutation - r-Combination Connection:

When order matters, the # of choices grows

Example: $\{A, B\}$ vs. (A, B) and (B, A).

But … grows by how much? There are $r!$ possible arrangements, so:

$$P(n,r) = \binom{n}{r} \cdot r!, \text{ or } \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Example:

$$\binom{5}{3} = \frac{P(5,3)}{3!} = \frac{60}{6} = 10$$

Or: $\dfrac{5!}{3!(5-3)!} = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 10$

r- Combination

Example:

From a Chess Club of 12 members, how many ‘traveling squads’ of 6 can be formed?

Answer: Order doesn't matter, so: $\binom{12}{6} = 924$

The University is forming a committee with 5 (of 9 available) faculty and (of 8) staff members. In how many ways can the committee be formed?

Answer: By combinations and the Multiplication Principle:

$$\binom{9}{5} \cdot \binom{8}{4} = 126 \cdot 70 = 8820$$

Repetition and Permutations

- We've already seen this!
  - **but we haven't been allowing repetition recently**

Example:

Recall: 3 digit octal numbers:

With repetition: $8 \cdot 8 \cdot 8$

Without repetition: $8 \cdot 7 \cdot 6$

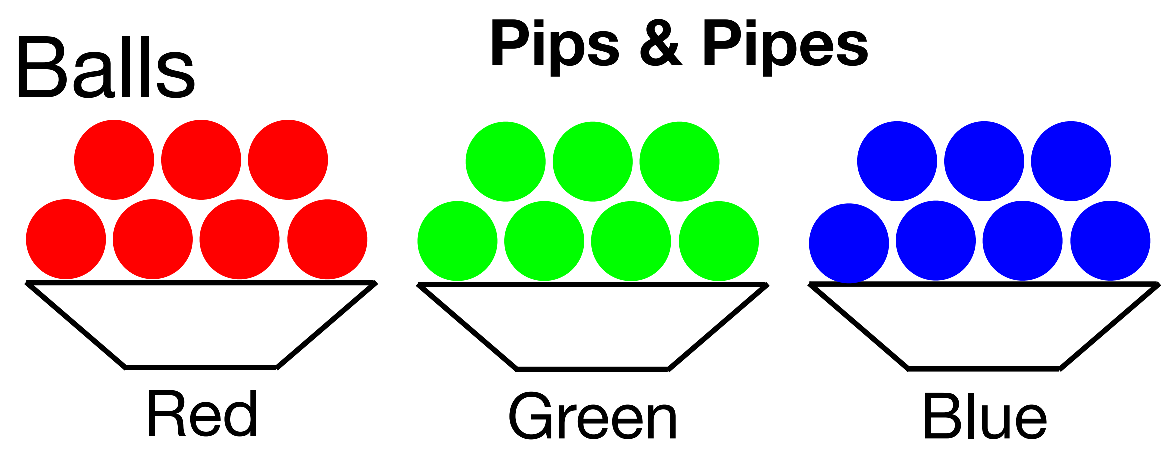
- In General: When object repetition is permitted, the number of r-permutations of a set on n objects is n^r

Here: 8^3

Repetition and Combinations

Example: 'Experienced' Golf Balls

In how many ways can a golfer select two balls



Answer: 6 (RR,GG,BB,RG,RB,GB)

Imagine a ball tray - only the balls and dividers matter!

$$\begin{array}{ll} 200 \rightarrow \cdot\cdot\,|| & 110 \rightarrow \cdot|\cdot| \\ 020 \rightarrow |\cdot\cdot & 101 \rightarrow \cdot||\cdot \\ 002 \rightarrow ||\cdot\cdot & 011 \rightarrow |\cdot|\cdot \end{array}$$

We have 4 positions for 2 balls ($\binom{4}{2}$) and 2 remaining positives for dividers ($\binom{2}{2}$). By M.P.: $\binom{4}{2}\binom{2}{2} = 6$

Repetition and Combinations

Example: At a cafeteria, how many ways exist to select 4 utensils from bins of forks, spoons, knives, & soup spoons?

Answer: 4 bins ⇒ 3 dividers, and

3 dividers + 4 utensils = 7 items

$$\therefore \binom{7}{4} = \binom{7}{3} = \frac{7!}{4!3!} = 35$$

- In General: When repetition is allowed, the number of r-combinations of a set on n objects is

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$ here $r = 4$ utensils and $n = 4$ bins

Repetition and Combinations

- A Small Extension:

Example: Consider a pot-luck with 5 platters of food.A child must have one serving from each platter but may have 3 more servings of anything. In how many ways can the child form 8 total servings?

Answer: Ignore the first 5 servings, there's just one way to select them. Then: 5 platters ⇒4 dividers, plus 3 servings = 7 items.

So, $\binom{7}{3} = 35$

- In General: When repetition is allowed, the number of r-combinations of a set on n elements when one of each is included in r is

$\binom{r-1}{r-n} = \binom{r-1}{n-1}$ here $r = 8$ servings and $n = 5$ platters

Another View of Repetition and Combinations

- Consider: An integer variable can represent the quantity of items selected with repetition

Example: The Golf Ball Problem (again!)

Let r, b, g be the numbers of red, blue, and green balls the customer selects. Clearly $r, b, g \in \mathbb{Z}$.

We need solutions of $r + b + g = 2$ where r, b, g are ≥ 0.

Or we need 2 pips (the sum) and 2 pipes (the plus signs).

Again, $\binom{4}{2} = 6$ ways to buy 2 golf balls of the 3 colors

Another View of Repetition and Combinations

Example: The Pot-luck Dinner Problem (again!)

Here, our equation is $x_1 + x_2 + x_3 + x_4 + x_5 = 8$ where $x_i \geq 1$. (≥ 1 b/c we need ≥ 1serving each.)

Pips and pipes needs each term to be ≥ 0 To achieve this, let $y_i = x_i - 1$. This transforms the equation to:

$$y_1 + y_2 + y_3 + y_4 + y_5 = 3 \text{ where } y_i \geq 0$$

Or we need 3 pips (the sum) and 4 pipes (the plus signs).

As before, $\binom{7}{3} = \binom{7}{4} = 35$ ways to get 3 servings.

Generalized Permutations

- Idea: What if some elements are indistinguishable?

Example:

Review: How many arrangements of the letters A-F are possible?

Answer: $6! = 720 = P(6,6)$

How many arrangements of A, A, and B are possible?

Answer: 3: AAB, ABA, BAA because the A's are indistinguishable. Otherwise, it's a simple permutation: $3! = 6$. The difference: There are $2! = 2$ ways to order the A's in each of the three arrangements, but here those orderings don't matter. Thus, $\frac{3!}{2!} = 3$

Generalized Permutations

- What if we have indistinguishable copies of multiple elements?

Example:

How many distinguishable arrangements of the letters in the word TATTOO are possible?

Answer: $\frac{6!}{3!2!} = 60$. There are 6! letter arrangements possible, but 3! arrangements of the T's and the $2!$ arrangements of the O’s don't matter.

In general: If we have *n* objects of *t* different types, and there are i_k indistinguishable objects of type *k*, then the number of distinct arrangements is $P(n; i_1, i_2, \ldots, i_t) = \frac{n!}{i_1!i_2!\ldots i_t!}$

Generalized Permutations

- We can view $P(n; i_1, i_2, ..., i_t)$ in terms of combinations

Example: Consider TATTOO again

There are $\binom{6}{3} = 20$ ways to place the T's, leaving 3 empty spaces. There are $\binom{3}{2} = 3$ ways to place the O's and $\binom{1}{1} = 1$ way to place the A. By the multiplication Principle: $\binom{6}{3}\binom{3}{1}\binom{1}{1} = 20 \cdot 3 \cdot 1 = 60.$

In General:

$$P(n; i_1, i_2, ..., i_t) = \binom{n}{i_1}\binom{n - i_1}{i_2}\binom{n - i_1 - i_2}{i_3}\cdots\binom{n - \ldots - i_{t-1}}{i_t}$$

More Fun With Combinations

- What if we created a table of $\binom{n}{k}$ values?

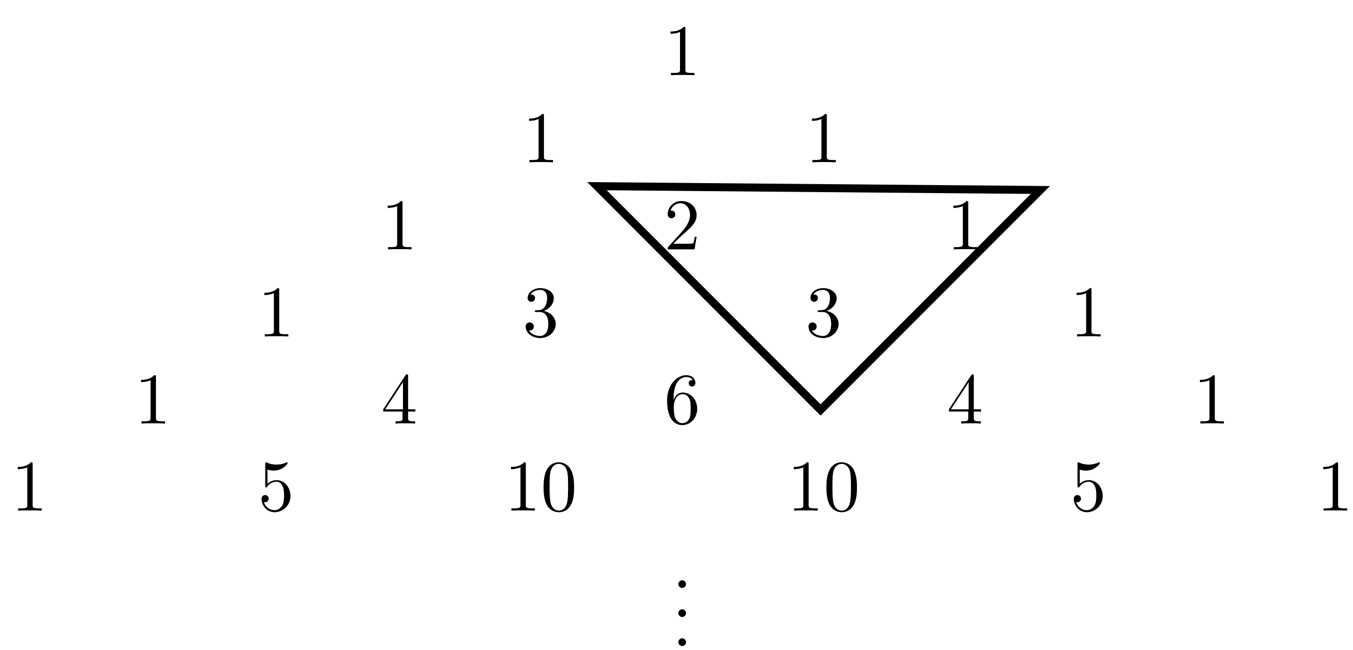
k

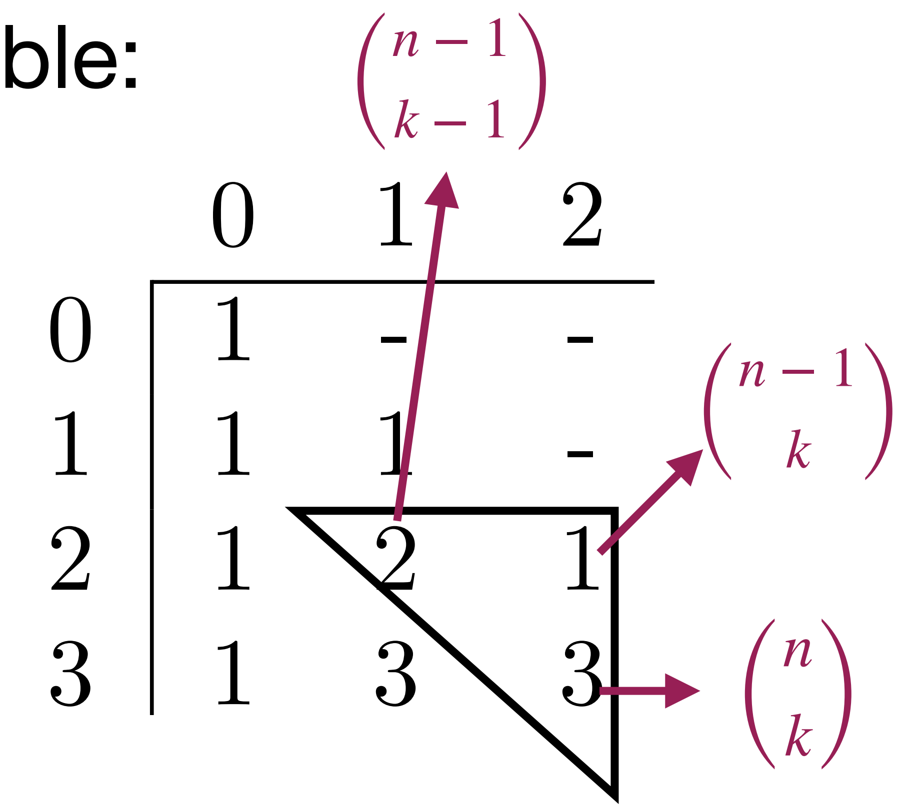
n	0	1	2	3	4	5
0	1	-	-	-	-	-
1	1	1	-	-	-	-
2	1	2	1	-	-	-
3	1	3	3	1	-	-
4	1	4	6	4	1	-
5	1	5	10	10	5	1

This should look familiar…

Pascal's Triangle

… is just the centered rows of the $\binom{n}{k}$ table:





Observations:

1. Each row is palindromic: $\binom{n}{k} = \binom{n}{n-k}$

2. "Pascal's Identity" (Inverted Triangles): $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

More Fun With Combinations

Conjecture: $\binom{n}{k} = \binom{n}{n-k}$**, where** $0 \le k \le n$

Proof (direct, algebraic):

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!}$$ **[By definition]**

$$= \frac{n!}{(n-k)!k!}$$ **[Simplified]**

$$= \binom{n}{k}$$ **[By definition]**

Therefore, $\binom{n}{k} = \binom{n}{n-k},\ 0 \le k \le n$

Pascal's Identity (Combinatorial Argument Example)

Conjecture: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, **where** $1 \le k \le n$

Consider $S = \{W, X, Y, Z\}$. $|S| = 4 = n + 1$. Let $k = 2$.

There are $\binom{n+1}{k} = \binom{4}{2} = 6$ subsets of S of size 2:

$\{\{W, X\}, \{W, Y\}, \{W, Z\}, \{X, Y\}, \{X, Z\}, \{Y, Z\}\}$

Consider element W. Either a subset contains W or it does not.

If W is included, to compete the subset we need one more item from the remaining three. There are $\binom{3}{1}$ such subsets.

If W is not included, to compete the subset we need two more items to make the subset, but again we have just three items to choose from: $\binom{3}{2}$

Thus the number of subsets is $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$ $(6 = 3 + 3)$

Pascal's Identity (Combinatorial Proof)

Definition: *Combinatorial Proof*

An argument based on the principles of counting

Conjecture: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, **where** $1 \leq k \leq n$

Proof (direct, combinatorial ("double counting")):

Let $d \in D$, and $|D| = n + 1$. Because sets are unordered, there are $\binom{n+1}{k}$ subsets of D of size k.

Some of these subsets include d, and the rest do not.

(Continued….)

Pascal's Identity (Combinatorial Proof)

Case 1: Subsets that include d. Differences are due to the other $k-1$ elements. We need to select those elements from the remaining (that is, non-d) values of D.

There are $\binom{n}{k-1}$ ways to do this.

Case 2: Subsets not including d. We need to select k more elements from D, again not counting d. There are $\binom{n}{k}$ ways to do this.

Together this is the total quantity of subsets.

Therefore, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ where $1 \leq k \leq n$

The Binomial Theorem

The values of Pascal's triangle appear in numerous places.

For instance:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= 1a + 1b \\
(a+b)^2 &= 1a^2 + 2ab + 1b^2 \\
(a+b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3
\end{aligned}$$

Generalize this, and you've got the Binomial Theorem.

The Binomial Theorem

Theorem: $(a+b)^n = \sum_{k=0}^{n} \left[\binom{n}{k} \cdot a^{n-k} \cdot b^k \right]$

Proof: See Rosen Sect 6.4 p 437-8. (Combinatorial!)

Example: Find the coefficient of x^5y^3 in the expansion of $(x+y)^8$.

By the above theorem: $k = 3, \; n = 8$, and so the coefficient is $\binom{8}{3} = 56$