
Finite Probability

Probability

Definition: Probability

The probability that a specific event will occur is the ratio of the number of occurrences of interest to the number of possible occurrences

- The occurrences of interest are called events (E).
- The set of possible occurrences is the Sample space (S).
- These are finite sets, hence the term *finite* probability
- The occurrence probability of an interest event:

$$p(E) = \frac{|E|}{|S|}$$

Probability

Please note: (a) $\forall e \in S, p(e) > 0$ (b) $\sum_{e \in S} p(e) = 1$

Example:

The probability of drawing a 10 from a 52-card deck:

$$p(E) = \frac{|E|}{|S|} = \frac{|\{10C, 10D, 10H, 10S\}|}{52} = \frac{1}{13}$$

Roll a pair of 6-sided, fair dice. What is the probability of one showing a 4 and the other either a 1 or a 6?

$$E = \{(4,1), (4,6), (1,4), (6,4)\}; \quad |E| = 4$$

$$S = \text{all possible rolls of two dice}; \quad |S| = 36$$

$$\text{Thus: } p(E) = \frac{|E|}{|S|} = \frac{4}{36} = \frac{1}{9}$$

Applications of Counting to Probability

1. Probability of Winning the Powerball Lottery

To play: Choose 5 of 59 'white' and 1 of 35 'red' (Powerball) numbers. # of ways to play?

There are $\binom{59}{5}$ ways to choose the whites, and
 $\binom{35}{1}$ ways to choose a red.

By the Multiplication Principle, there are

$$\binom{59}{5} \cdot \binom{35}{1} = 175,223,510 \text{ ways to play}$$

$$p(\text{win}) = \frac{|E|}{|S|} = \frac{1}{175,223,510} \approx 5 \times 10^{-9} = 0.000000005$$

Applications of Counting to Probability

2. Principle of Inclusion-Exclusion

$$\text{Recall: } |E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example:

54 students: 28 freshman (13 men, 15 women)
20 sophomores (12 men, 8 women)
6 juniors (5 men, 1 women)

The probability of drawing a woman's name:

$$p(\text{woman}) = \frac{15 + 8 + 1}{54} = \frac{24}{54} = \frac{4}{9} = 0.\bar{4}$$

... of drawing either a woman (W) or a freshman (F)

$$\begin{aligned} p(W \cup F) &= p(W) + p(F) - p(W \cap F) \\ &= \frac{24}{54} + \frac{28}{54} - \frac{15}{54} = \frac{37}{54} = \left(1 - \frac{17}{54}\right) (\text{fraction of male non-freshman}) \end{aligned}$$

Probabilistic Reasoning

- Each drawer of a 3x2 dresser holds either a red or a blue T-shirt. One row of drawers has two red shirts, one row has two blue, and one row has one of each. You open one drawer and see a red T-shirt. What is the probability that the shirt in the other drawer is the same row is also red?

Probabilistic Reasoning

- One solution approach: Enumerate the possibilities.
WLOG:

Dresser		Open Drawer Containing	Shirt Color in Other Drawer?
R1	R2	R1	Red
B	B	R2	Red
R3	B	R3	Blue

$$p(\text{The opposite drawer has red}) = \frac{2}{3}$$

Bertrand's Box Paradox (1889)

Probabilistic Reasoning

- Suppose you are on a game show, and you're given the choice of 3 doors: Behind one door is a car; behind the others goats. You pick a door, say No. 1, and the host who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you “Do you. Want to pick door No. 2?” Is it to your advantage to switch your choice?

From “Ask Marilyn”, Parade, Sept. 9, 1990.

- Reference:
 - www.marilynvossavant.com/articles/gameshow.html

Care to Play?

URL: <http://dcity.org/brain-games/three-door-dilemma>

Probabilistic Reasoning

- But ... why? Three views:

1. Enumerate the Possibilities

WLOG, assume that we always pick door #1

There are 3 possibilities:

Case	Door #1	Door #2	Door #3	Host Reveals	Switch?
(a)	Car	Goat	Goat	#2 or #3	You lose!
(b)	Goat	Car	Goat	#3	You win!
(c)	Goat	Goat	Car	#2	You win!

$$\therefore p(\text{win}) = \frac{2}{3} \text{ if you switch.}$$

Probabilistic Reasoning

2. Car / Not Car

Consider a partitioning of the doors into two groups: Selected and Unselected

$$p(\text{Car behind Selected}) = \frac{1}{3} \text{ and}$$

$$p(\text{Car behind unselected}) = \frac{2}{3}$$

After the host reveals a goat, those probabilities have not changed!

$$p(\text{win without switching doors}) = \frac{1}{3} \text{ (stay with Selected)}$$

$$p(\text{win after switching doors}) = \frac{2}{3} \text{ (move to Unselected group)}$$

⇒ The remaining closed door represents the Unselected group

**Extra Slides (not
covered)**

Conditional Probability

Example:

If we have one fair 6-sided die:

$$p(6) = \frac{1}{6} \quad p((6,6)) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Let E_1 be rolling a 6 on the 1st roll, & E_2 a 6 on the 2nd

There are 6 ways for each to occur, but what if E_1 has *already* occurred?

In that case: $p(E_2)$ still is $\frac{1}{6}$ because the first rolls does not influence the second.

But ... what if the first does influence the second?

Conditional Probability

Definition: Conditional Probability

Let X and Y be events. The conditional probability of X given Y , denoted $p(X | Y)$, is $\frac{P(X \cap Y)}{p(Y)}$

Example:

Continuing the students example, what is $P(W | F)$?

$$\begin{aligned} p(W | F) &= p(W \cap F) / P(F) \\ &= \frac{15}{54} / \frac{28}{54} = \frac{15}{28} = 0.536\dots \end{aligned}$$

(Easy to see from 2 slides ago: 15 women of 18 freshman)

$$\text{(Recall that } p(W) = \frac{24}{54} = \frac{4}{9} = 0.\bar{4}\text{)}$$

Independence of Events

$$\text{Recall: } p(A | B) = \frac{p(A \cap B)}{p(B)}$$

Definition: *Independent*

Events A and B are independent if $p(A | B) = p(A)$

Example:

You flip a coin twice. The first flip comes up heads (E_1) and the second tails (E_2). Are these events independent?

$$P(E_1) = \frac{|\{(H, H), (H, T)\}|}{|\{(H, H), (H, T), (T, T), (T, H)\}|} = \frac{1}{2} \quad (p(E_2) = \frac{1}{2}, \text{ too}).$$

$$p(E_1 \cap E_2) = \frac{|\{(H, T)\}|}{4} = \frac{1}{4}$$

$$p(E_2 | E_1) = \frac{p(E_2 \cap E_1)}{p(E_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = p(E_2) \quad \therefore \text{independent}$$

Independence of Events

Example:

Events: $E = \{2,4,6\}$, $H = \{1,2,3\}$ $T = \{1,2,3,4\}$

Are E and H independent?

$$p(E) = \frac{1}{2}, \quad p(H) = \frac{1}{2}, \quad p(T) = \frac{2}{3}, \quad p(E \cap H) = \frac{1}{6}$$

$$p(E | H) = \frac{p(E \cap H)}{p(H)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \neq p(E)$$

$\therefore E$ and H are not independent!

Well ... why not??

(Continued ...)

Independence of Events

$$E = \{2,4,6\}, H = \{1,2,3\} \quad T = \{1,2,3,4\}$$

In our situation of $p(E | H)$, we are given that only 1, 2, or 3 was rolled. Only one of those shared with E , thus $p(E | H) = \frac{1}{3}$.

However, without the limitation of H , a roll will be a member of E without probability $\frac{1}{2}$.

This shows that the probability of E occurring changes if H is assumed to have occurred first. Thus, E is dependent on H , or E and H are not independent.

Note: E and T are independent:

$$p(E \cap T) = \frac{1}{3}, \text{ and so } p(E | T) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} = p(E)$$

Probabilistic Reasoning

Another approach to the door problem using conditional probability

3. Conditional Probability (Still assuming we pick door #1)

$$\text{Recall: } p(X|Y) = \frac{p(X \cap Y)}{p(Y)}, \text{ or } P(Y) \cdot p(X|Y) = p(X \cap Y)$$

Let C_i be the event that the car is behind door i , and

Let H_k be the event of the host opening door k

$$\begin{aligned} p(\text{win of switch}) &= p(H_3 \cap C_2) + p(H_2 \cap C_3) \\ &= p(C_2) \cdot p(H_3 | C_2) + p(C_3) \cdot p(H_2 | C_3) \\ &= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 \\ &= \frac{2}{3} \end{aligned}$$

(Note the application of the Addition Principle!)