Functions

2.3

Functions as Relations

- Consider $f(x) = x + 1, x \in \mathbb{Z}$
  - Alternate notation:
    - $f = \{(x, x + 1) \,|\, x \in \mathbb{Z}\}$

Definition: *Function*

A function f from set X to Y, denoted $f : X \to Y$, is a relation from X to Y, where $f(x)$ is defined $\forall x \in X$ and, if $(x, y) \in f$, then y is the **only value** returned by $f(x)$.

Functions as Relations

Example:

$f = \{(x, x+1) \mid x \in \mathbb{Z}\}$

- **Is f a relation?**
- **Is $f(x)$ is defined for all integers?**
- **Is $x + 1$ is the only value returned by $f(x)$?**

Letter grades assigned to students:

- $X = \{$**Zeus, Leto, Apollo**$\}$
- $Y = \{A, B, C, D, E\}$
- $G = \{($**Zeus**$, A), ($**Leto**$, A), ($**Apollo**$, C)$

Playposit

Definition: *Function*

A function f from set X to Y, denoted $f : X \rightarrow Y$, is a relation from X to Y, where $f(x)$ is defined $\forall x \in X$ and, if $(x, y) \in f$, then y is the **only value** returned by $f(x)$.

Function Terms

Let $f : X \to Y$ be a function and let $f(n) = p$.

- X is the ___domain___ of f
- Y is the ___codomain___ of f
- f ___maps___ X to Y
- p is the ___image___ of n
- n is the ___pre-image___ of p
- f's ___range___ is the set of all images of X's elements

Note: A function's range need note equal its codomain.

Function Terms

Example:

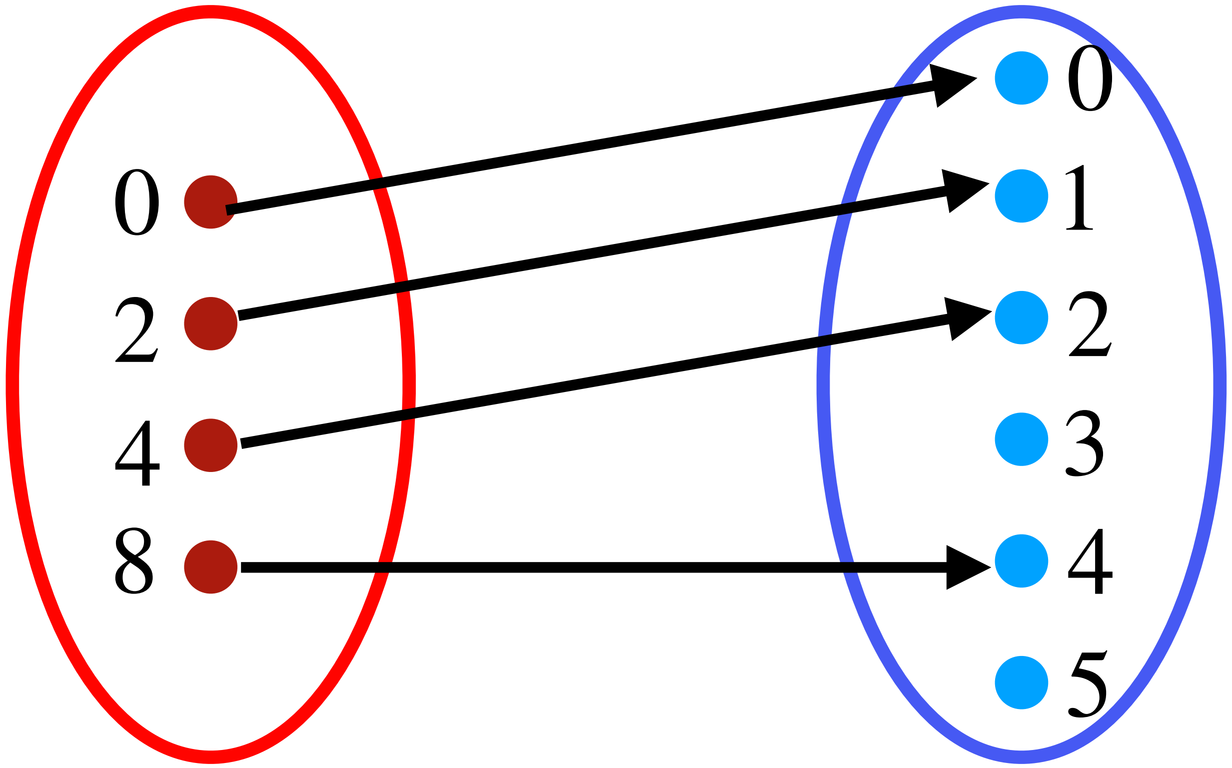
$g = \{(a, b) \mid b = a/2\},\ a \in \{0,2,4,8\},\ b \in \{0,1,2,3,4,5\}$

- Domain: $\{0,2,4,8\}$
- Codomain: $\{0,1,2,3,4,5\}$
- Image (of $(8,4)$): 4
- Pre-image (of $(8,4)$): 8
- Range (of g): $\{0,1,2,4\}$

Digraph Representation

Example:

$$g = \{(a,b) \mid b = a/2\},\ a \in \{0,2,4,8\},\ b \in \{0,1,2,3,4,5\}$$



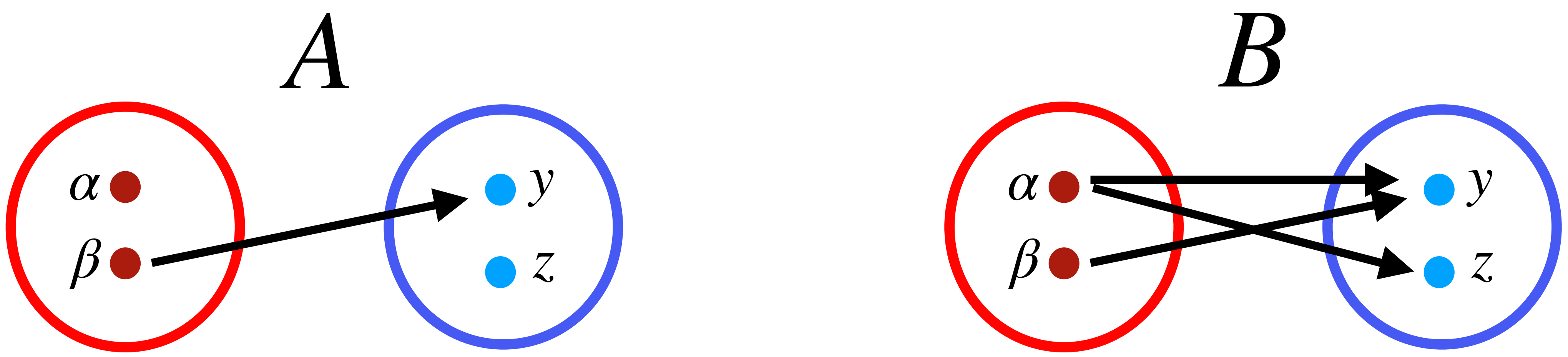
The incoming arrows identify the range members

Digraph Representation

Example:

$A = \{(\beta, y)\}$ from $\{\alpha, \beta\}$ to $\{y, z\}$

$B = \{(\alpha, y), (\alpha, z), (\beta, y)\}$ from $\{\alpha, \beta\}$ to $\{y, z\}$



Neither are functions:

In A, α is unused (function not defined $\forall$ domain)

In B, α is related to multiple codomain values

Two Functions You Need to Know

1. Floor ($\lfloor x \rfloor$)

Definition: *Floor Function*

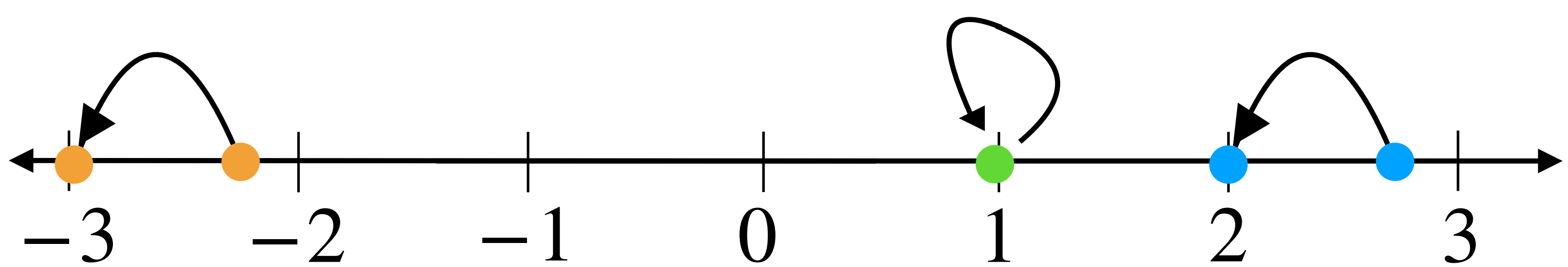
The floor of a real value n, denoted $\lfloor n \rfloor$, is the largest integer $\leq n$

Example:

$\lfloor 2.8 \rfloor = 2$

$\lfloor 1 \rfloor = 1$

$\lfloor -2.2 \rfloor = -3$



Always move left on the number line

See also: `Math.floor()` in the Java API

Two Functions You Need to Know

1. Floor ($\lfloor x \rfloor$) (cont)

Using Floor for Rounding to the Nearest integer

Easy: Just add 0.5, then 'floor it'. Example:

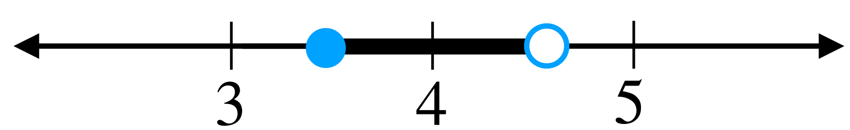
Round 3.50:

Round 3.99:

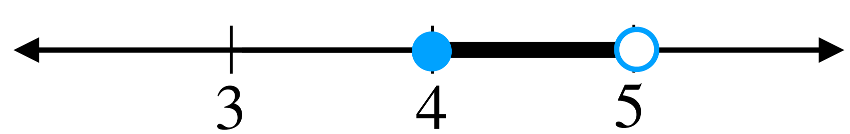
Round 4.49:

In General:

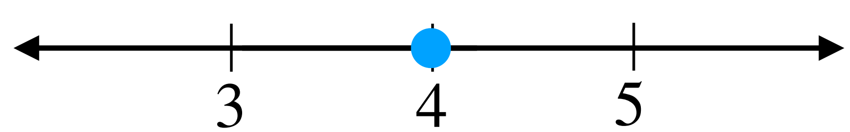
Range that rounds to 4



After 0.5 is added



After floor is applied



Two Functions You Need to Know

2. Ceiling ($\lceil x \rceil$)

Definition: [*Ceiling Function*]

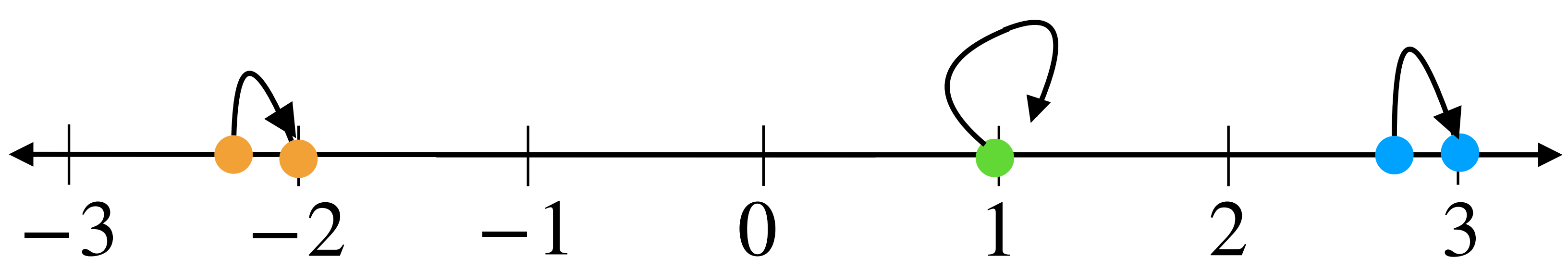
The ceiling of a real value m, denoted $\lceil m \rceil$, is the largest integer $\geq m$

Example:

$\lfloor 2.8 \rfloor = 2$

$\lfloor 1 \rfloor = 1$

$\lfloor -2.2 \rfloor = -3$



Always move right on the number line

See also: `Math.ceil()` in the Java API

Two Functions You Need to Know

2. Ceiling ($\lceil x \rceil$)

Example:

Plan: \$0.50 for calls ≤ 10 min., plus \$0.05 per additional minute

Example: 11.5 minute call ⇒ 60 cents.

First try:

Cost (length) $50 + 5 \cdot \lceil \text{length} - 10 \rceil$ cents.

⇒But: Fails when length ≤ 10

Fixed:

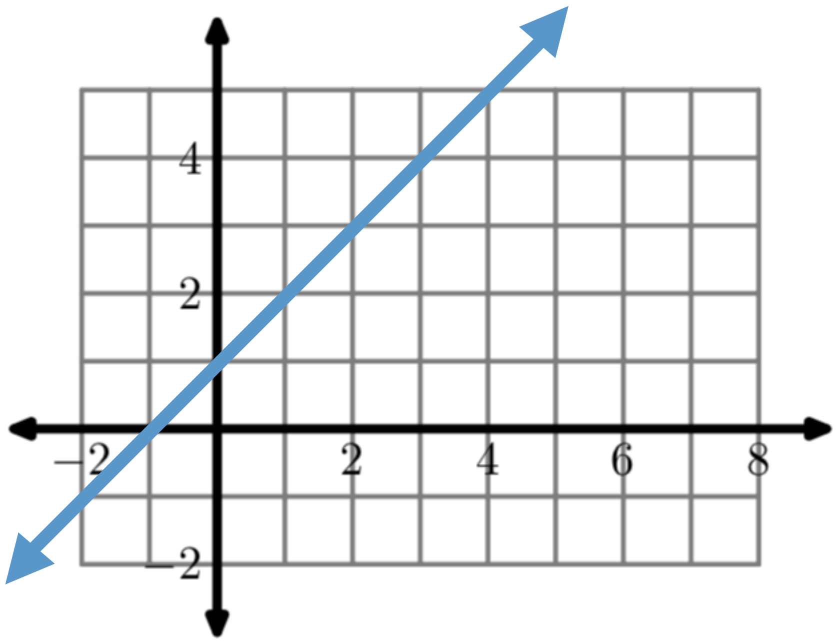
$$\text{Cost (length)} = \begin{cases} 50 & \text{length} \leq 10 \\ 50 + 5 \cdot \lceil \text{length} - 10 \rceil & \text{Otherwise} \end{cases}$$

Example: Type A UPC code Check Digits

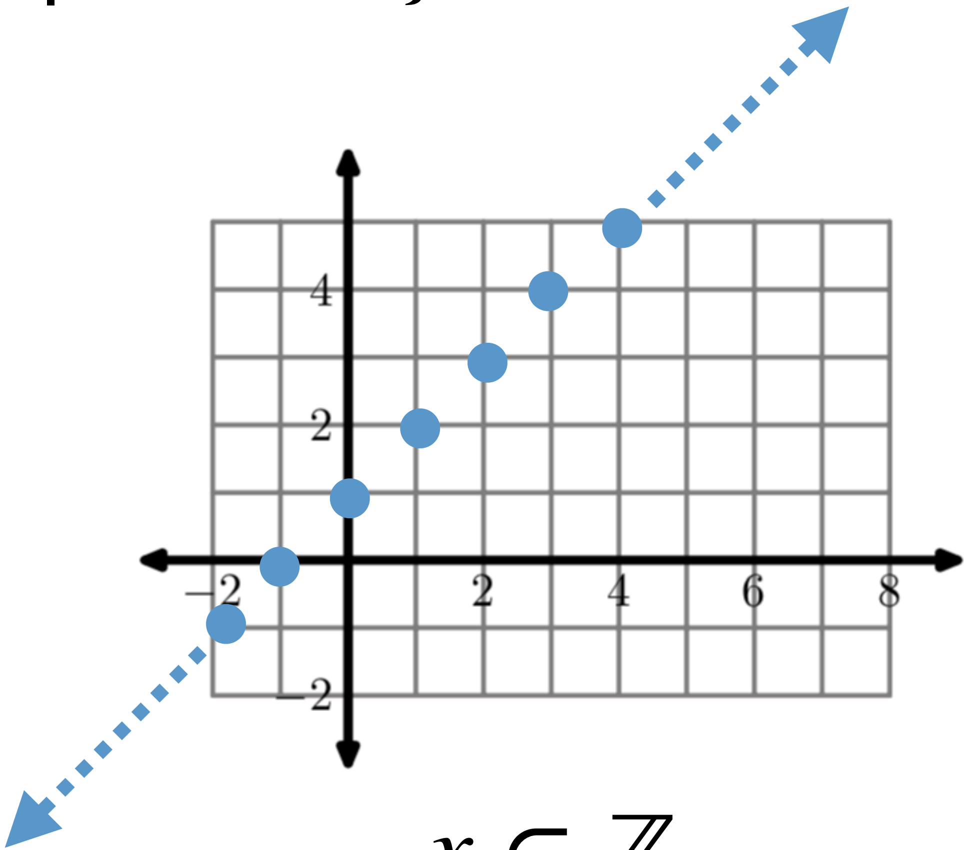
- The check digit equals the image of this function:
  - $s =$ Sum of digits in positions $1, 3, 5, 7, 9, \& \ 11$
  - $t =$ Sum of digits in positions $2, 4, 6, 8, \& \ 10$
  - $u = 3s + t$; the check digit is $(10 - u \,\%\, 10) \,\%\, 10$
- Using the above sample:
  - $s = 39$, $t = 24$, and $u = 3(39) + 24 = 141$
  - Check digit = $(10 - 141 \,\%\, 10) \,\%\, 10 = 9$

Plots of Functions

- Important Distinction: *Continuous* vs. *Discontinuous Functions*
- Consider: $f = \{(x, x+1) \mid x \in \ldots\}$



$x \in \mathbb{R}$

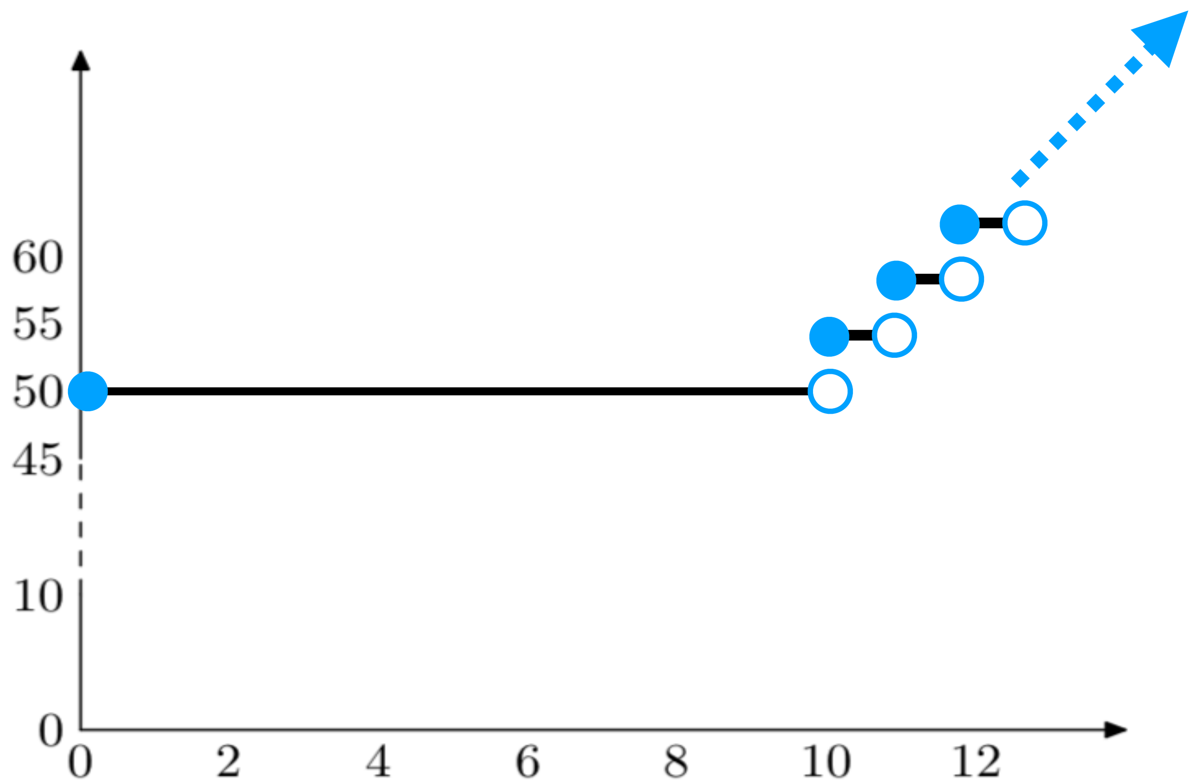


$x \in \mathbb{Z}$

Plots of Functions

- How should the plot of our long-distance calling plan function look?

$$\text{Cost (length)} = \begin{cases} 50 & \text{length} \leq 10 \\ 50 + 5 \cdot \lceil \text{length} - 10 \rceil & \text{Otherwise} \end{cases}$$



This is an example of a *piecewise function*

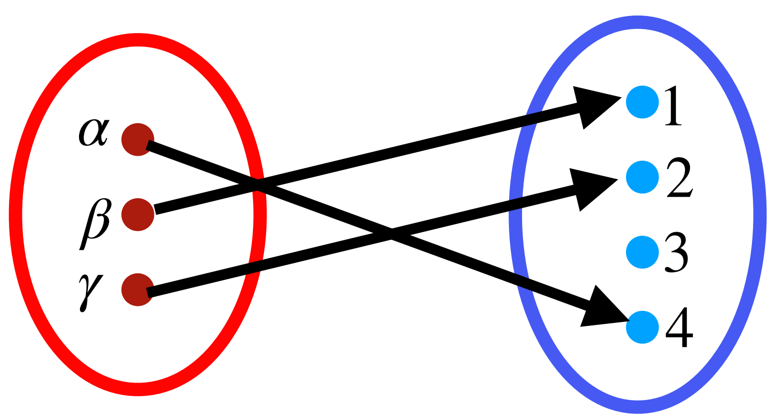
Categories of Functions: Injective

Definition: *Injective Functions* **(a.k.a One-to-One)**

A function f from set X to Y is injective if, for each $y \in Y, f(x) = y$ for at most one member of X

Example:

$F = \{(\alpha,4),(\beta,1),(\gamma,2)\}$ **from** $\{\alpha,\beta,\gamma\}$ **to** $\{1,2,3,4\}$



Each y has 0 or 1 incoming arrows

∴ it's injective

Categories of Functions: Surjective

Definition: *Surjective Functions* **(a.k.a Onto)**

A function f from set X to Y is surjective if, f's range is Y (that is, the range equals the codomain)

Example:

$F = \{(\alpha,4),(\beta,1),(\gamma,2)\}$ **from** $\{\alpha,\beta,\gamma\}$ **to** $\{1,2,3,4\}$

F **is <u>not</u> surjective: 3 is not used.**

Surjective functions: Each y **has** ≥ 1 **incoming arrows**

Categories of Functions: Bijective

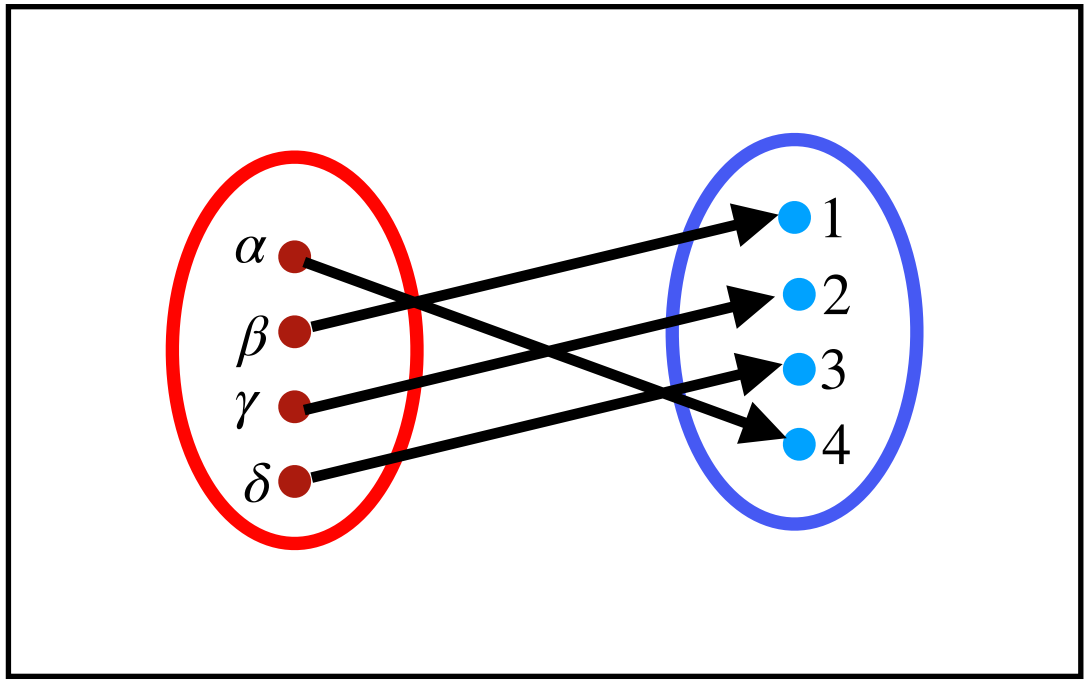
Definition: *Bijective Functions* **(a.k.a One-to-One Corresondence)**

A function f from set X to Y is bijective if it is both injective and surjective

Example:

Each y has exactly 1 incoming arrow

Note: $|X| = |Y|$



Odds and Ends

Definition: *Functional Composition*

Let $f : Y \to Z$ and $g : X \to Y$. The composition of f and g, denoted $f \circ g$, is the function $h = f(g(x))$, where $h : X \to Z$

Definition: *Inverse Functions*

The inverse of a bijective function f, denoted f^{-1}, is the relation $\{(y, x) \mid (x, y) \in f\}$

(Note the bijective requirement, otherwise the definition is the same as that of relational inverse)

Beyond Unary Functions

Definition: *Binary Function*

A binary function is a function $f : X \times Y \to Z$, $(f(x, y) = z)$

Example: Wind Chill (*)

$$WC(T, V) = (0.2175T - 35.75)V^{0.16} + 0.6215T + 35.74$$

where V is wind speed (mph) and T is air temp in ˚F

(Heat index is also a binary function, but messier!)

(*) Developed by the Joint Action Group for Temperature Indices and adopted by the US, UK, and Canada in Nov. 2001