
Proof by Induction

5.1

Review: Inductive and Deductive Reasoning

Definition: Inductive Argument

An argument that moves from specific observations to general conclusions

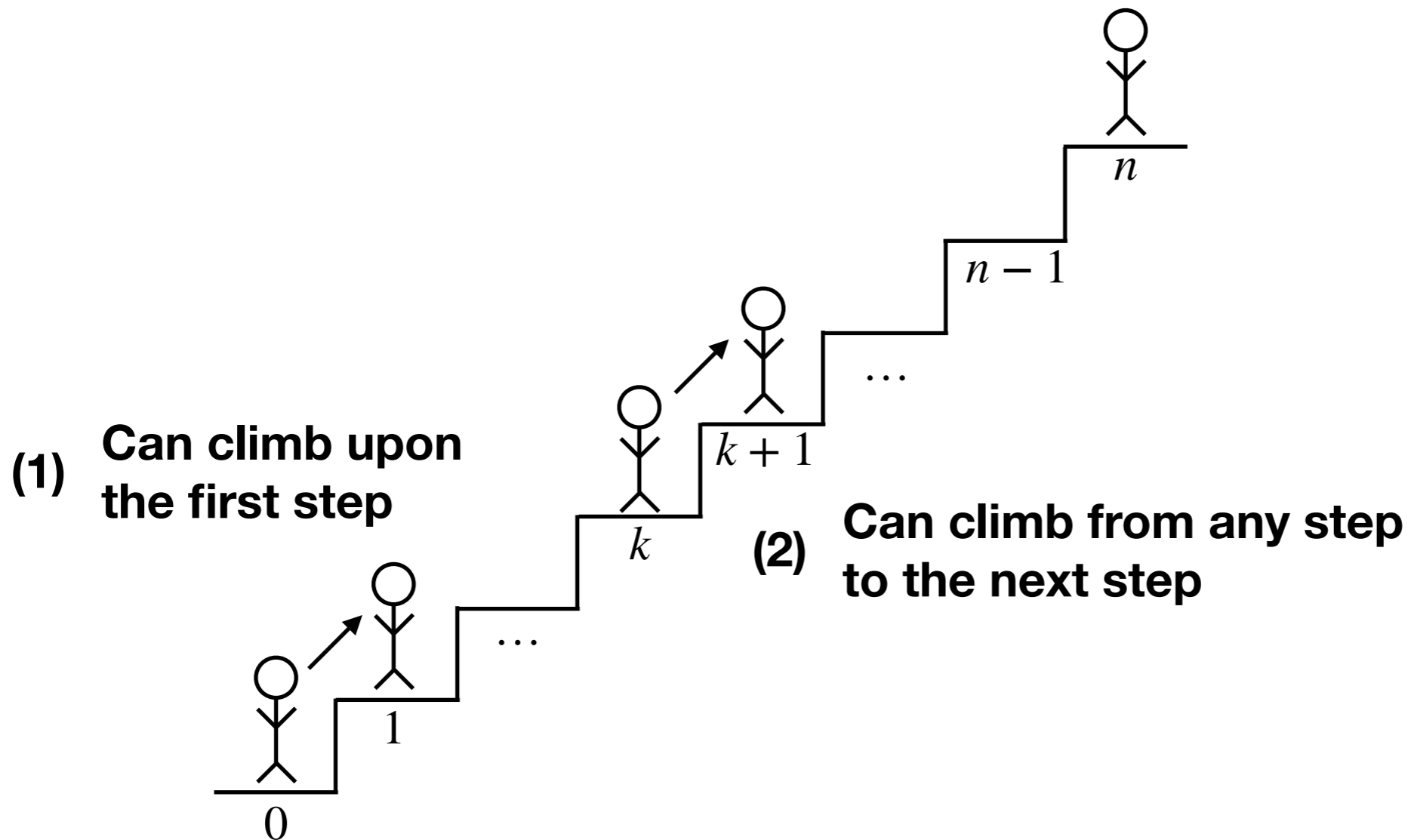
Definition: Deductive Argument

An argument that uses accepted general principles to explain a specific situation

Old Example: 3, 5, and 7 are prime numbers. Therefore all positive integers above 1 are prime numbers

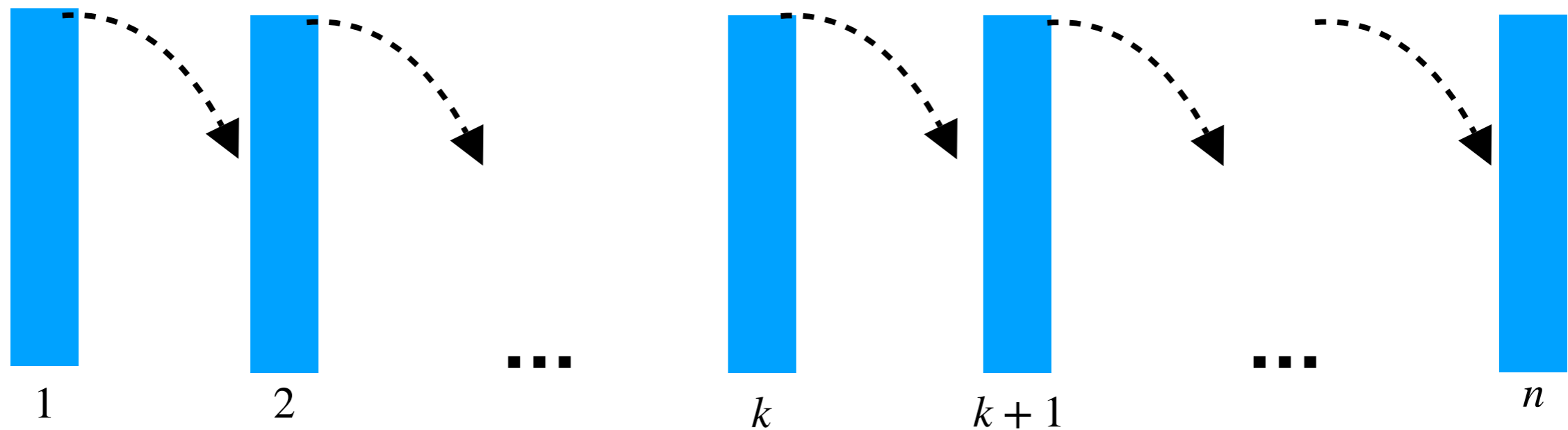
The Big Idea

Example: Climbing Stairs



The Big Idea

Example: Dominoes



(1) 1st domino must fall onto the 2nd

(2) Domino k must fall onto domino $k + 1$

But ... How Does Induction *Prove* Anything?

After all, you could fall off the stairs, space dominoes too far apart, etc. \Rightarrow Integers are more predictable!

Consider \mathbb{Z}^+ . If we can show that:

(a) A property holds for 1, and

(b) If the property holds for $n \in \mathbb{Z}$, then it also holds for the next integer ($n + 1$),

Then we can conclude that the property will hold for 2 (b/c $1 \rightarrow 2$), 3 (b/c $2 \rightarrow 3$), etc. - that is, it will hold for all positive integers.

\Rightarrow This is the *First Principle of Mathematical Induction*

The First Principle of Mathematical Induction

(often called weak induction)

Definition: First Principle of Mathematical Induction

If: (i) $P(a)$ is true for the starting point $a \in \mathbb{Z}^+$ and

(ii) (for any $k \in \mathbb{Z}^+$ $a \leq k$)

if $P(k)$ is true then $P(k + 1)$ is true

Then: $P(n)$ is true for all $n \in \mathbb{Z}^+$, $n \geq a$.

(i) is the *Basis Step*, and (ii) is the *Inductive Step*

In the $p \rightarrow q$ form:

$$[P(a) \wedge \forall k (P(k) \rightarrow P(k + 1))] \rightarrow \forall n P(n)$$

(Note that there's a proof within the proof!)

The Second Principle of Mathematical Induction

(often called strong induction)

Definition: Second Principle of Mathematical Induction

If: (i) $P(a)$ is true for the starting point $a \in \mathbb{Z}^+$ and
(ii) (for any $k \in \mathbb{Z}^+, a \leq k$)
If $P(j)$ is true for all $j \in \mathbb{Z}^+$ such that $a \leq j \leq k$
then $P(k + 1)$ is true

Then: $P(n)$ is true for all $n \in \mathbb{Z}^+, n \geq a$.

Note the difference:

Strong \Rightarrow All $P(i)$ from $P(a)$ through $P(k)$ are assumed true

Weak \Rightarrow Just $P(k)$ is assumed true

But ... Which One Should I Use?

- Whichever is appropriate! (if both are, use the easier!)
- Why? Because Weak Induction \equiv Strong Induction
 - (1) Easy to see that Strong \rightarrow Weak
 - (2) Weak \rightarrow Well-Ordering \rightarrow Strong

(See Rosen 8/e Sec. 5.2 p. 361-2 and Ex. 41-43)

Definition: *The Well-Ordering Property:*

Every non-empty set of positive integers has a smallest member

Example #1: A summation

Conjecture:
$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Proof (by Weak Induction):

Basis Step (base case):

Let $n = 1$.
$$\sum_{i=1}^1 i = 1 = \frac{1(1+1)}{2}, \text{ Ok!}$$

Inductive Step:

If
$$\sum_{i=1}^k i = \frac{k(k+1)}{2},$$
 then
$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Inductive Hypothesis

(Continues...)

Example #1: A summation

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \sum_{i=1}^k i + (k + 1) \\ &= \frac{k(k + 1)}{2} + (k + 1) \text{ [By the Inductive Hypothesis]} \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{(k + 1)(k + 2)}{2}\end{aligned}$$

Thus, $\sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2}$

Example #2: An Inequality

Conjecture: $n^2 > n + 1, \forall n \geq 2, n \in \mathbb{Z}^+$

Proof (by Weak Induction):

Basis: Let $n = 2$. $2^2 > 2 + 1$ (or $4 > 3$)? Yes!

Inductive Step: If $k^2 > k + 1$ then $(k + 1)^2 > k + 2$

$$(k + 1)^2 = k^2 + 2k + 1$$

By Inductive hypothesis, we know that $k^2 > k + 1$.

Thus $(k^2) + 2k + 1 > (k + 1) + 2k + 1 = 3k + 2$

(Continues ...)

Example #2: An Inequality

We still must show that $3k + 2 \geq k + 2$. Let's simplify that.

By subtracting $k + 2$ from both sides gives $2k \geq 0$

Dividing both sides by 2 shows that $k \geq 0$ is all that needs to be shown

But we already know that $k \geq 0$ is true, because we were given that $k \geq 2$.

Thus, $(k + 1)^2 > k + 2$

Therefore $k^2 > k + 1, \forall k \geq 2$

Example #2: An Inequality

Here's an alternative inductive step, starting with the I.H.:

Inductive: If $k^2 > k + 1$, then $(k + 1)^2 > k + 2$

- | | |
|-------------------------------------|--|
| (1) $k^2 > k + 1$ | [Given (Inductive Hypothesis)] |
| (2) $k^2 + 2k + 1 > k + 1 + 2k + 1$ | [Add $2k + 1$ to both sides] |
| (3) $(k + 1)^2 > 3k + 2$ | [Algebra] |
| (4) $k > 0$ | [Given ($k \geq 2 \rightarrow k > 0$)] |
| (5) $2k > 0$ | [Multiply both sides by 2] |
| (6) $3k + 2 > k + 2$ | [Add $k + 2$ to both sides] |
| (7) $(k + 1)^2 > k + 2$ | [(3),(6),Transitivity] |

Problem: Very 'magical': Hard to see that adding $2k + 1$ and assuming $k > 0$ are the right things to do!

The Basis Step Only Seems Pointless

Conjecture: $\sum_{i=0}^{n-1} 2^i = 2^n, \forall n \geq 1$

Proof (by Weak Induction):

Basis: $n = 1, \sum_{i=0}^0 2^i = 2^1$ (or $2^0 = 2^1$)

But: The inductive step still works!

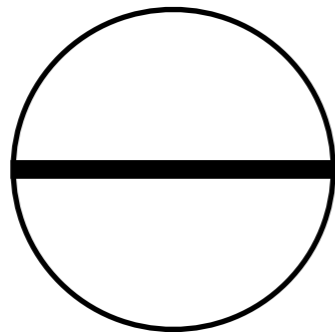
Inductive Step: if $\sum_{i=0}^{k-1} 2^i = 2^k$ then $\sum_{i=0}^k 2^i = 2^{k+1}$

$$\sum_{i=0}^k 2^i = \sum_{i=0}^{k-1} 2^i + 2^k = 2^k + 2^k \text{ [by the I.H]} = 2(2^k) = 2^{k+1}$$

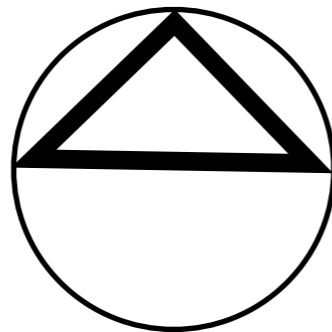
What is provable: $\sum_{i=0}^{k-1} 2^i = 2^k - 1$

Some “Obvious” Patterns That Aren’t

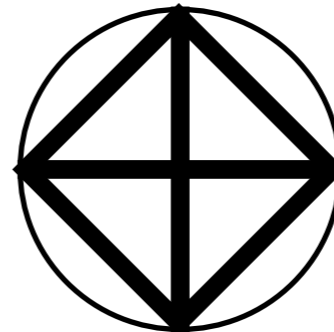
1. Circle Division by Chords



2 points
2 regions



3 points
4 regions



4 points
8 regions

p	regions
2	2
3	4
4	8
\vdots	\vdots
p	$2^{p-1}?$

No! $p = 6$ has just 31 regions

The correct expression:

$$\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

$$= 1 + \binom{n}{2} + \binom{n}{4}$$

(Note: No three chords may intersect at the same point.)

Some “Obvious” Patterns That Aren’t

2. Is $n^2 - n + 41$ prime $\forall n \geq 1$

No! It must fail at $n = 41$ ($41^2 - 41 + 41$ is clearly composite). However, all members of the sequence $\{n^2 - n + 41\}_{n=1}^{40}$ are prime.

3. Is $991n^2 + 1$ never a perfect square $\forall n \geq 1$?

The 1st n that makes $991n^2 + 1$ a perfect square is

12,055,735,790,331,447,442,538,767

Strong Induction

- Review:
 - In *Strong Induction*, $P(a)$ must be true and, for any $k \geq a$, if $P(a) \wedge P(a + 1) \wedge \dots \wedge P(k - 1) \wedge P(k)$ is true, then $P(k + 1)$ is true.

Weak needs only the truth of the preceding case to show the truth of the next case

Strong needs the truth of multiple preceding cases.

\Rightarrow Only use those preceding cases that are required by your situation!

Example: Strong Induction

Conjecture: Let $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$. Also assume $a_k = a_{k-1} + a_{k-2} + a_{k-3}$. Is $a_n \leq 2^n$, $\forall n \geq 3$?

Proof (by Strong Induction):

Basis: We start w/ $n = 3$ b/c $n = 0, 1, 2$ are given.

$$a_3 = a_2 + a_1 + a_0 = 3 + 2 + 1 = 6 \leq 2^3 = 8. \text{ Ok!}$$

Inductive Step: if $a_i \leq 2^i$, $\forall i$ $3 \leq i < k$, then $a_k \leq 2^k$.

$$\text{Begin with } a_k = a_{k-1} + a_{k-2} + a_{k-3}$$

(continues...)

(Note, that the IH goes to $k-1$ instead of k .)

Example: Strong Induction

By the Inductive Hypothesis we know that $a_{k-1} \leq 2^{k-1}$, $a_{k-2} \leq 2^{k-2}$, and $a_{k-3} \leq 2^{k-3}$.

It follows that $a_k \leq 2^{k-1} + 2^{k-2} + 2^{k-3}$.

In turn, this is $\leq 2^{k-1} + 2^{k-2} + \dots + 2^0$.

In a prior example, we claimed that is the sum equal to $2^k - 1$.

Combining these expressions, we see that $a_k \leq 2^k - 1$ which means that $a_k \leq 2^k$

Therefore, $a_n \leq 2^n$, $\forall n \geq 3$

Structural Induction

- Simple Idea: Apply induction (weak or strong) to “structures”, such as:
 - Strings
 - Binary trees
 - Program statements
 - \vdots

Ex: In a complete binary tree,
 $\# \text{ leaf nodes} = \# \text{ internal nodes} + 1$