Matrices

Section 2.6

Why are We Studying Matrices?

- Matrices have plenty of uses in CS
- Representation …
  - … of the graph data structure
  - … of functions and relations (next two topics we'll cover)
- Affine transformations in Computer Graphics
  - Example to come!

Matrix Fundamentals

Definition: *Matrix*

A matrix is an n-dimensional collection of values

Notation

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 4 & 3 \\ 0 & 2 \\ 1 & \boxed{1} \end{bmatrix} \end{array} = \begin{pmatrix} 4 & 3 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

a 3×2 matrix

$a_{3,2}$

$$A = [a_{ij}] = (a_{ij})$$

Matrix Fundamentals

Definition: *Square Matrices*

Matrices in which the number of rows equals the number of columns

$$\begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$$

a 2×2 square matrix

Matrix Fundamentals

Definition: *Square Matrices*

Matrices in which the number of rows equals the number of columns

Definition: *Matrix Equality*

Matrices A and B are equal if they share the same dimensions and each pair of corresponding elements is equal, i.e. $a_{ij} = b_{ij}$ for $1 \leq i \leq n, 1 \leq j \leq m$

Matrix Fundamentals

Definition: *Transposition*

The transposition of an $m \times n$ matrix A is an $n \times m$ matrix A^T in which the rows and columns are exchanged. $a_{ij} = a^T_{ji}$

$$A = \begin{bmatrix} 4 & 3 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 4 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Matrix Fundamentals

Definition: *Transposition*

The transposition of an $m \times n$ matrix A is an $n \times m$ matrix A^T in which the rows and columns are exchanged. $a_{ij} = a_{ji}^T$

Definition: *Matrix Symmetry*

Matrix A is symmetric if $A = A^T$ (note: A is square).

$$\begin{bmatrix} 4 & 3 & 1 \\ 3 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Matrix Operations

1. Matrix Addition

Definition: *Matrix Addition* (a.ka. Matrix Sum)

The sum of two $n \times m$ matrices A and B is the $n \times m$ matrix C such that $c_{ij} = a_{ij} + b_{ij}$

$$A = \begin{bmatrix} 6 & 0 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 + -1 & 0 + 3 \\ 4 + 1 & 2 + 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 5 & 2 \end{bmatrix}$$

Note: $A + B = B + A$
(matrix addition is commutative).

Matrix Operations

2. Scalar Product

Definition: *Scalar*

A scalar is a real number (in this context)

Definition: *Scalar Product*

The product of a scalar d and an $n \times m$ matrix A is the $n \times m$ matrix B such that $b_{ij} = d \cdot a_{ij}$

$$A = \begin{bmatrix} 6 & 0 \\ 4 & 2 \end{bmatrix} \qquad \frac{1}{2}A = \begin{bmatrix} \frac{1}{2} \cdot 6 & \frac{1}{2} \cdot 0 \\ \frac{1}{2} \cdot 4 & \frac{1}{2} \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$$

Matrix Operations

3. Matrix Product

Definition: *Matrix Product (a.ka. Matrix Multiplication)*

The product of an $m \times n$ matrix A and an $n \times k$ matrix B, is an $k \times m$ matrix $C = A \cdot B$ in which $c_{ij} = \sum_{k=1}^{n} (a_{ik} \cdot b_{kj})$.

- Matrix multiplication is associative and distributive

Matrix Operations

Recall: $c_{ij} = \sum_{k=1}^{n} (a_{ik} \cdot b_{kj})$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$m \times k$ $\qquad$ $k \times n$ $\qquad$ $m \times n$

Let $C = A \cdot B$ **Element** c_{ij} **is calculated by:**

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{ik} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \end{bmatrix} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \dots + a_{ik} \cdot b_{kj}$$

Matrix Operations

Recall: $c_{ij} = \sum_{k=1}^{n} (a_{ik} \cdot b_{kj})$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} 7 & 1 \\ 9 & 2 \end{bmatrix}$$

Because A has 2 columns and B has 2 rows, AB can be computed

Boxes example: Row [1 3] and column [4 1]: $1 \cdot 4 + 3 \cdot 1 = 7$

$$BA = \begin{bmatrix} 6 & 13 \\ 1 & 3 \end{bmatrix}$$

Matrix product is **<u>not</u>** generally commutative

Matrix Operations

$$A = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \end{bmatrix} \qquad A \cdot B = \begin{bmatrix} \square & \square \\ \square & \square \\ \square & \square \end{bmatrix}$$

3×1 $\qquad$ 1×3

$$B \rightarrow \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$A \rightarrow \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2 & -1 \\ 6 & 3 \end{bmatrix} = A \cdot B$$

Matrix Operations

$$A = \begin{bmatrix} 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$1 \times 3 \qquad\qquad 3 \times 1$$

$$A \cdot B = [0 \cdot 2 + -1 \cdot 1 + 3 \cdot 3] = [8]$$

Identity Matrix

- Remember the concept of Multiplicative Identity?

  - $1 \cdot x = x$

Definition: *Identity Matrix*

The Identity Matrix is an $n \times n$ matrix (I_n) populated with 1's down the main (upper left to lower right) diagonal and with 0's elsewhere.

$$\text{If } A \text{ is } m \times n : \underset{m \times n}{A} \cdot \underset{n \times n}{I_n} = \underset{m \times m}{I_m} \cdot \underset{m \times n}{A} = \underset{m \times n}{A}$$

Matrix Power

Definition: *n^{th} Matrix Power*

The n^{th} power of a $m \times m$ matrix A, denoted A^n, is the result of $n-1$ successive matrix products of A

$$A^4 = ((A \cdot A) \cdot A) \cdot A = A \cdot (A \cdot (A \cdot A))$$

$$A^0 = ?$$

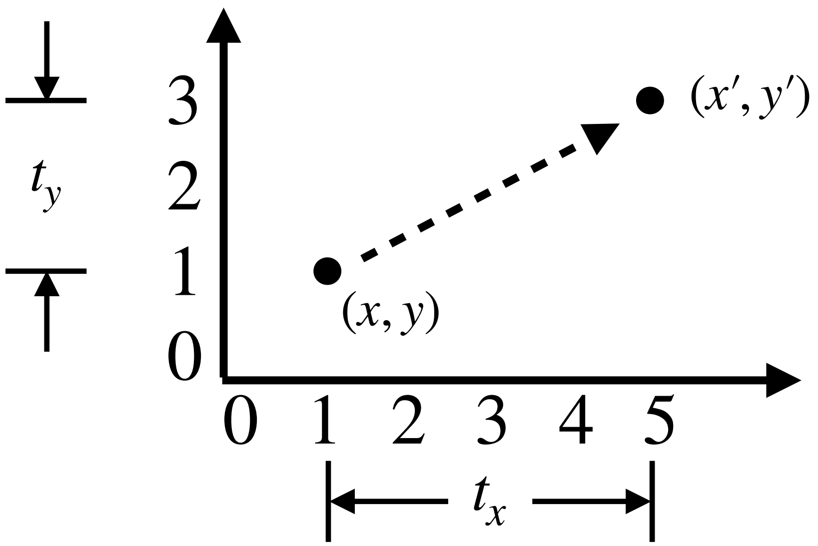
[Answer: $A^0 = I_m$**, because** A **is** $m \times m$**]**

Example: Affine Transformations

- Used to 'move' objects in computer graphics
- Background:

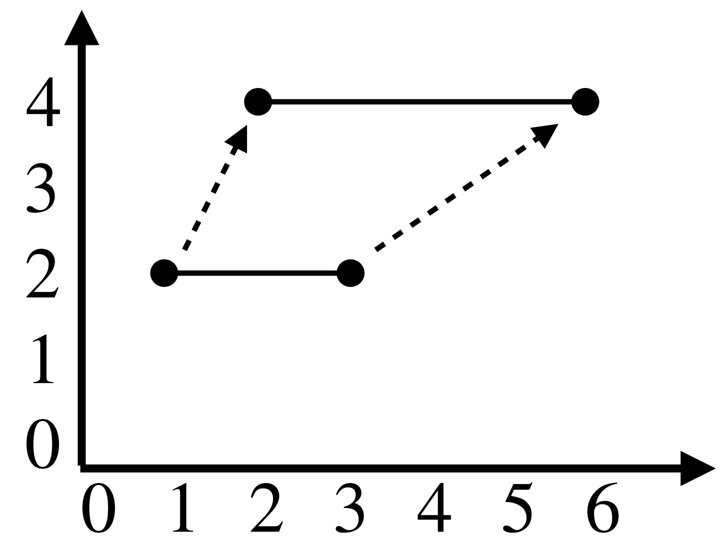
<u>**Translation:**</u> $(x, y) \Rightarrow (x', y')$

$$x' = x + t_x$$
$$y' = y + t_y$$

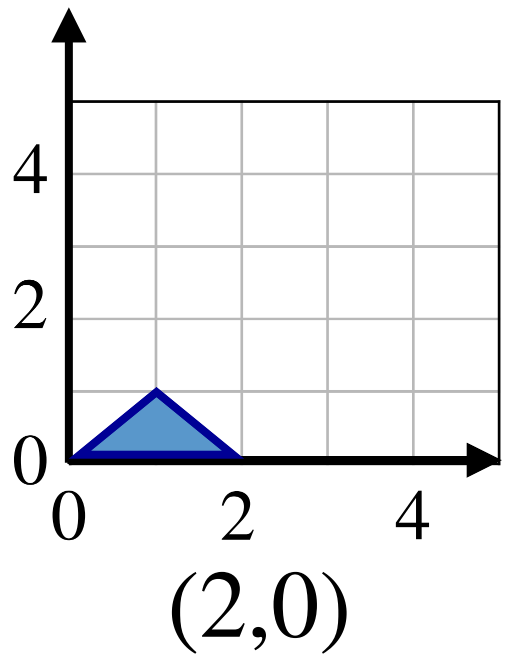


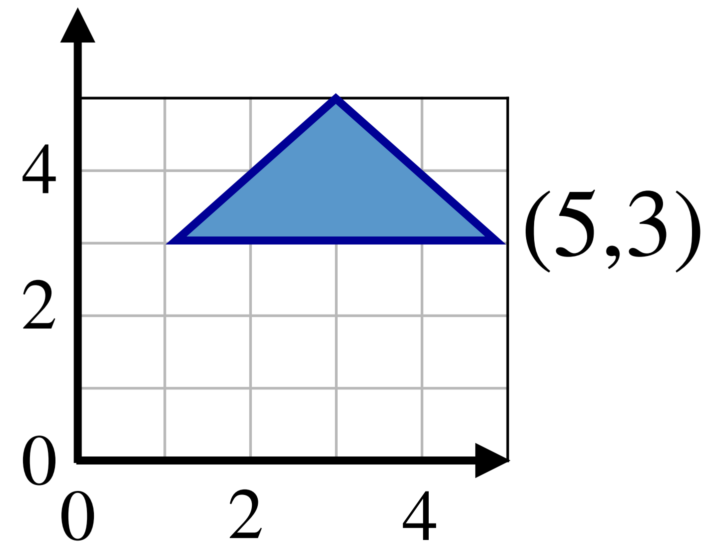
<u>**Scaling:**</u> $(x, y) \Rightarrow (x', y')$

$$x' = s_x \cdot x$$
$$y' = s_y \cdot y$$



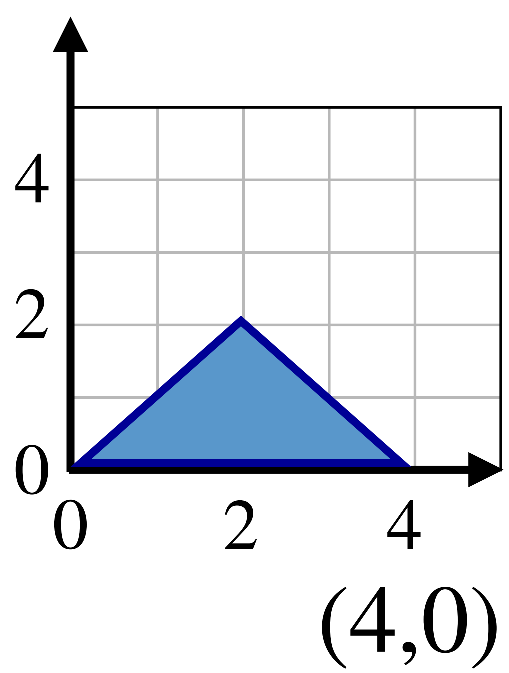
Example: Affine Transformations





(1) Scale (2x2)

$s_x = s_y = 2$



(2) Translate

$t_x = 1; \;\; t_y = 3$

Example: Affine Transformations

Translation: $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$ **Scaling:** $\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So: $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(Combined transformation matrix ↗)

$$(2,0) \Rightarrow (5,3) : \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

Zero-One Matrices

- All entries are 0 or 1, used to represent discrete structures
- Three Operations:

1. 'Join': $(A \vee B)$ - inclusive OR of pairs of values
2. 'Meet': $(A \wedge B)$ - AND of corresponding pairs

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M \vee N = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M \wedge N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Zero-One Matrices

1. 'Boolean Product': Consider $A(m \times n)$ and $B(n \times l)$. $C = A \odot B$ is $m \times l$ where

$$c_{ij}. = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad G = E \odot F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$g_{33} = \bigvee (e_{3k} \wedge f_{k3}) = (1 \wedge 0) \vee (1 \wedge 1)$$

$$= 0 \vee 1 = 1$$

Zero-One Matrices

Definition: r^{th} Boolean Power

The r^{th} Boolean Power of an $n \times n$ matrix A, $A^{[r]}$, is the $n \times n$ resulting form of $r - 1$ successive boolean products.

Note: $A^{[0]} = I_n$