Quantification

From last time:

Definition: <u>*Proposition*</u>

A declarative sentence that is either true (**T**) or false (**F**), but not both.

- Tucson summers never get above 100 degrees
- If the doorbell rings, then my dog will bark.
- You can take the flight if and only if you bought a ticket
- But <u>NOT</u> *x* < 10

Why do we want use variables?

- Propositional logic is not always expressive enough
- Consider: "All students like summer vacation"
- Should be able to conclude that "If Joe is a student, he likes summer vacation".
- Similarly, "If Rachel is a student, she likes summer vacation" and so on.
- Propositional Logic does not support this!

Predicates

Definition: *Predicate (a.k.a. Propositional Function)*

A statement that includes at least one variable and will evaluate to either **true** or **false** when the variables(s) are assigned value(s).

• Example:

 $S(x) : (-10 < x) \land (x < 10)$ E(a, b) : a eats b

• These are not complete!

Predicates

Definition: <u>Domain (a.ka. Universe) of Discourse</u>

The collection of values from which a variable's value is drawn.

• Example:

 $S(x) : (-10 < x) \land (x < 10), x \in \mathbb{Z}$

E(a,b): *a* eats *b*, $a \in$ People, $b \in$ Vegetables

- In This Class: Domains may NOT hide operators
 - OK: Vegetables
 - Not-OK: Raw Vegetables (Vegetable ∧ ¬Cooked)

Predicates

$S(x): (-10 < x) \land (x < 10), \ x \in \mathbb{Z}$

E(a,b) : *a* eats *b*, $a \in$ People, $b \in$ Vegetables

- Can evaluate predicates at specific values (making them propositions):
 - What is *E*(*Joe*, *Asparagus*)?
 - What is S(0)?

Combining Predicates with Logical Operators

- In *E*(*a*, *b*) : *a* eats *b*, *a* ∈ people, *b* ∈ vegetables,
 change the domain of *b* to "raw vegetables".
 - E(a,b) : a eats b, $a \in people$, $b \in vegetables$
 - C(b): *b* is cooked, $b \in$ vegetables
- Combining the two:
 - $E(a,b) \land \neg C(b), a \in \text{people}, b \in \text{vegetables}$

Quantification

- Idea: Establish truth of predicates over sets of values.
- Two common generalizations:
 - <u>Universal Quantification</u> ($\forall x P(x), x \in D$)
 - Considers all values from the domain of discourse
 - Existential Quantification ($\exists x S(x), x \in D$)
 - Considers one or more values form the domain of discourse

Note: Do <u>not</u> use the books non-standard $\exists x$ notation ("uniqueness quantifier", Rosen 8/e p.46)

Quantifications in Propositional Logic

- Universal Quantification
 - $\forall x P(x), x \in D \equiv P(d_0) \land P(d_1) \land P(d_2) \land \dots$

- Existential Quantification
 - $\exists x S(x), x \in D \equiv S(d_0) \lor S(d_1) \lor S(d_2) \lor \dots$

Universal Quantification:

Universal quantification of P(x), $\forall x P(x)$, is the statement "P(x) holds for all objects x in the domain of discourse"

 $\forall x P(x)$ is true only when P(x) is true for **every** x in the domain, and false otherwise.

- Example 1:
 - $Q(x) : x = x^2, x \in \mathbb{R}$
 - $\forall x Q(x), x \in \{-1, 0, 1\}$?

- Q(-1) = False, $-1 \neq (-1)^2$
- A value *x* for which P(x) is false is a counterexample $\forall x P(x)$

Universal Quantification:

Universal quantification of P(x), $\forall x P(x)$, is the statement "P(x) holds for all objects x in the domain of discourse"

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- Example 2:
 - $P(x, y) : x + y \text{ is even}, x, y \in \mathbb{Z}$
 - $\forall x \forall y \ P(x, y), x, y \in \mathbb{Z}^{odd}$ True! (Easy to prove)

• Existential Quantification:

Existential quantification of P(x), $\exists x P(x)$, is "There exists an element x in the domain of discourse such that P(x)"

 $\exists x P(x)$ is true if **at least one** element *x* in the domain such that P(x) is true

- Example 1:
 - $Q(x) : x = x^2, x \in \mathbb{R}$
 - $\exists x Q(x), x \in \{-1, 0, 1\}$?

Q(0) = True, $0 \neq 0^2$

• Existential Quantification:

Existential quantification of P(x), $\exists x P(x)$, is "There exists an element x in the domain of discourse such that P(x)"

 $\exists x P(x)$ is true if **at least one** element *x* in the domain such that P(x) is true

- Example 2:
 - $P(x, y) : x + y \text{ is even}, x, y \in \mathbb{Z}$
 - $\exists x \exists y \ P(x, y), x, y \in \mathbb{Z}^{odd}$

True! Universal quantifier covers existential

Examples: Converting from English to Quantified Predicates

• Consider this conversational English statement:

Every actor in The Office is great.

• How can we express that statement in logic notation?

• Consider this conversational English statement:

Every actor in The Office is great.

• How can we express that statement in logic notation? Let P(x): Actor x is great, $x \in \text{People}$ $\forall x \ P(x), x \in \text{People}$

Stilted English: For every person *x*, *x* is a great actor. **Conversationally**: All people are great actors.

PROBLEM: This is not quite the desired meaning

IDEA: Let's focus the domain!

• Attempt #2: Every actor in The Office is great.

Let P(x): Actor x is great, $x \in$ **People who act in The Office**

$\forall x \ P(x), x \in \mathbf{People who act in the Office}$

Stilted English: For every person x who acts in The Office, x is a great actor.Conversationally: All actors in the Office are great.

PROBLEM: The domain has a hidden predicate

IDEA: Let's create a new predicate.

• Attempt #3: Every actor in The Office is great.

Let P(x): Actor x is great, $x \in \mathbf{People}$

Let Q(x) : x was in The Office, $x \in$ People

 $\forall x \ (P(x) \land Q(x)), x \in \mathbf{People}$

- **Stilted English**: For every person *x*, *x* is in The Office and *x* is a great actor.
- **Conversationally**: All people are in the Office and are great.

PROBLEM: If true, implies that <u>all people</u> are in The Office!

IDEA: Try a different compound predicate

• Attempt #4: Every actor in The Office is great.

Let P(x): Actor x is great, $x \in \text{People}$ Let Q(x) : x was in The Office, $x \in \text{People}$ $\forall x \ (Q(x) \to P(x)), x \in \text{People}$

Stilted English: For every person *x*, if *x* was in The Office, then *x* is a great actor.
Conversationally: Every actor in the Office is great.
PROBLEM: Isn't *P*(*x*), really two predicates in one?

IDEA: Break it apart

[Why not $P(x) \rightarrow Q(x)$? That says all great actors are in the office]

• Attempt #5: Every actor in The Office is great.

Let P(x): x is an actor, $x \in \text{People}$ Let A(x): x is a great actor, $x \in \text{People}$ Let Q(x) : x was in The Office, $x \in \text{People}$ $\forall x ((Q(x) \land P(x)) \rightarrow A(x)), x \in \text{People}$

Stilted English: For every person *x*, if *x* is an actor and was in The Office, then *x* is a great actor.

Conversationally: Every actor in the Office is great.

--SUCCESS! --

(This is the version to learn!)

Implicit Quantification

- The "all" can be implicit in the English statement
- Example:
 - Adding an odd # to itself produces an even #

O(x) : x is odd, $x \in \mathbb{R}$

 $E(x) : x \text{ is even}, x \in \mathbb{R}$

 $\forall x \; (O(x) \to E(x+x)), \; x \in \mathbb{Z}$

 $\forall x \; (O(x) \to \overline{O(x+x)}), \, x \in \mathbb{Z}$

Note the implicit \forall is implicit in the sentence

Example: Existential Quantification

• Consider this conversational English statement:

At least one breed of dog is cute.

How can we express that statement in logic notation?

Let C(x) : x is cute, $x \in \text{Dog Breeds}$

 $\exists x \ C(x), x \in \text{Dog Breeds}$

English: There is at least one dog breed x such that x is cute.

Example: Existential Quantification

• Express this more specific statement in logic:

Some of the large fluffy dog breeds are cute. Let L(x) : x is large, $x \in Dog$ Breeds Let F(x) : x is fluffy, $x \in Dog$ Breeds Let C(x) : x is cute, $x \in Dog$ Breeds

 $\exists x (L(x) \land F(x) \land C(x)), x \in \text{Dog Breeds}$

These alternatives don't work! Why? $\exists x (L(x) \land F(x)) \rightarrow C(x), x \in \text{Dog Breeds}$ $\exists x (L(x) \land C(x)) \rightarrow F(x), x \in \text{Dog Breeds}$

Example: Existential Quantification

• Express this statement in logic:

Every last one of the large fluffy dog breeds are cute.

Let L(x) : x is large, $x \in \text{Dog Breeds}$

Let F(x) : x is fluffy, $x \in \text{Dog Breeds}$

Let C(x) : x is cute, $x \in \text{Dog Breeds}$

 $\forall x [(L(x) \land F(x)) \rightarrow C(x)], x \in \text{Dog Breeds}$

If a dog is both large and fluffy, then it is cute. (vs. ... $\wedge C(x)$: **All** dog breeds are large, fluffy and cute

Typically \forall pairs with \rightarrow and \exists goes with \land

Nested Quantifiers

- Sometimes we may need to use multiple quantifiers
 - We can't express "Everybody loves someone" using a single quantifier.
 - Suppose predicate loves(x, y): "Person x loves person y"

Nested Quantifiers

- loves(x, y): "Person x loves person y"
- The four possible nestings:



Evaluating Nested Quantifiers of the Same Type

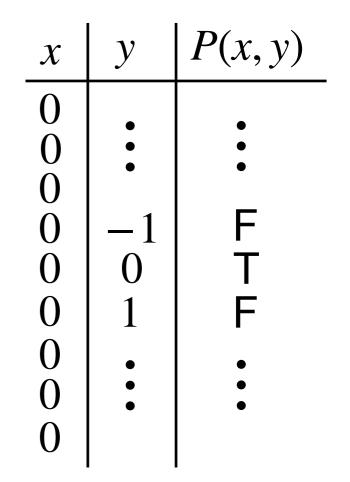
- Example: **loves**(*x*, *y*): "Person *x* loves person *y*"
 - $\forall x \forall y \ \mathbf{loves}(x, y)$
 - "Everyone loves everyone".
 - $\exists x \exists y \text{ loves}(x, y)$
 - "There is someone who loves someone else" (or possibly themself!)

- Example: **loves**(*x*, *y*): "Person *x* loves person *y*"
 - $\exists x \forall y \ \mathbf{loves}(x, y)$
 - "There is someone who loves everyone"
 - $\forall x \exists y \ \mathbf{loves}(x, y)$
 - "Everyone loves at least one person (possibly themself!)"

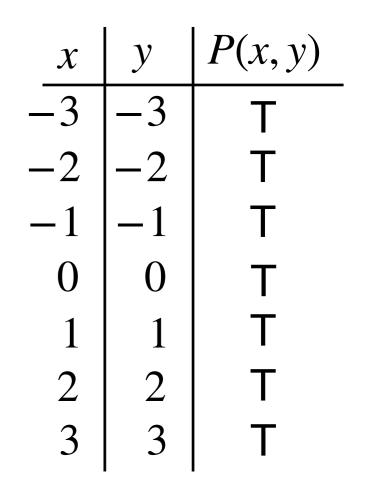
- Distinguishing $\exists x \forall y \ S(x, y)$ from $\forall i \exists k \ T(i, k)$
 - $\exists x \forall y \ S(x, y)$ "There exists an x such that, for every y, S(x, y) is true."
 - Somewhere in *x*'s domain is an *x* that can be paired with any *y*'s domain to make *S*(*x*, *y*) true.
 - $\forall i \exists k \ T(i,k)$ "For any *i* there exists a *k* such that T(i,k) is true.
 - No matter which *i* is selected, we can find some *k* to pair with the *i* to make *T*(*i*, *k*) true. (Note that the *k* may vary with the *i*)

- Example:
 - Given $P(x, y) : x y = 0, x, y \in \mathbb{Z}$
 - Evaluate: $\exists x \forall y P(x, y)$

- Example:
 - Given $P(x, y) : x y = 0, x, y \in \mathbb{Z}$
 - Evaluate: $\exists x \forall y \ P(x, y) =$ False
 - (No such magical integer x exists!)



- Example:
 - Given $P(x, y) : x y = 0, x, y \in \mathbb{Z}$
 - Evaluate: $\exists x \forall y \ P(x, y) =$ False
 - Evaluate: $\forall x \exists y P(x, y) =$ **True**
 - (No matter the *x*, there's an integer *y* (y = x) that makes P(x, y) true.)



Negation of Quantified Expressions

• Remember De Morgan's Laws for Propositions? Well...

Definition: <u>Generalized De Morgan's Laws</u>

The Generalized De Morgan's Laws are the pair of equivalences:

$$\neg \forall x P(x) \equiv \exists \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall \neg P(x)$$

Demonstration: $\neg \forall x P(x) \equiv \exists x \neg P(x)$

• Reminder:

$$\forall x \, S(x), x \in D \equiv S(d_1) \land S(d_2) \land S(d_3) \dots$$

 $\exists x S(x), x \in D \equiv S(d_1) \lor S(d_2) \lor S(d_3) \dots$

Demonstration: $\neg \forall x P(x) \equiv \exists x \neg P(x)$

• Let S(x) : x is cute, $x \in D$. Let $D = \{a \mid dog breeds\}$

$$\forall x \, S(x), x \in D \equiv S(d_1) \land S(d_2) \land S(d_3) \dots$$

$$\neg \forall x \, S(x), x \in D \equiv \neg (S(d_1) \land S(d_2) \land S(d_3) \dots)$$

$$\equiv \neg S(d_1) \lor \neg S(d_2) \lor \neg S(d_3) \dots$$

$$\equiv \exists x \neg S(x), x \in D$$

$$\exists x S(x), x \in D \equiv S(d_1) \lor S(d_2) \lor S(d_3) \dots$$

Negation of Nested Quantifiers

- Apply De Morgan's Laws from left to right
- Example:

$$\neg (\exists x \forall y (G(x) \lor \neg H(y)))$$

$$\equiv \forall x \neg (\forall y (G(x) \lor \neg H(y))) \text{ [General DeMorgan]}$$

$$\equiv \forall x \exists y \neg (G(x) \lor \neg H(y)) \text{ [General DeMorgan]}$$

$$\equiv \forall x \exists y (\neg G(x) \land H(y)) \text{ [DeMorgan]}$$

 Consider the conversational (& correct!) English statement

Only one citizen of Montana is a member of the U.S. House of Representatives

• And consider this awkward but useful rewording:

A member of the US House of Representatives exists in the set of citizens of Montana, and, if anyone in Montana is a member of the House, that person is the representative

 That rewording is useful because it can be directly expressed logically:

R(x): x is a member of the US House of Representatives, $x \in \mathbf{People}$

 $\exists x(R(x) \land \forall y[R(y) \rightarrow (y = x)]), x, y \in \textbf{Citizens of Montana}$

This domain should be simplified, but using it makes the logic easier to read (for now)

 That rewording is useful because it can be directly expressed logically:

R(x): x is a member of the US House of Representatives, $x \in \mathbf{People}$

 $\exists x(R(x) \land \forall y[R(y) \rightarrow (y = x)]), x, y \in \text{Citizens of Montana}$

- Interpretation: (At least one) \land (No more than one)
- .: Impossible for there to be two representatives!

• Key observation:

Exactly 2 \equiv At least 2 \land At most 2 (n = 2) \equiv ($n \ge 2$) \land ($n \le 2$)

• Awkward English:

At least two citizens of Montana are U.S. Senators, and at most two citizens of Montana are U.S. Senators.

• Better:

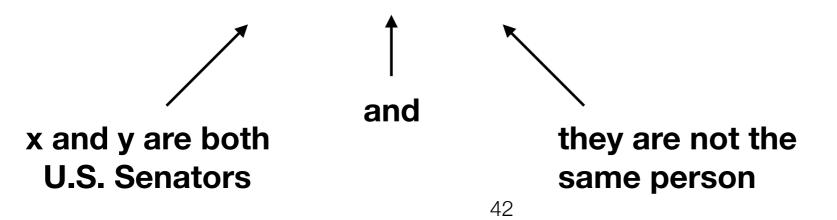
Exactly two citizens of Montana are U.S. Citizens

- Consider the two halves separately:
 - 1. "At least two citizens of Montana are U.S. Senators"

2. "At most two citizens of Montana are U.S. Senators"

Expressing "At least two"

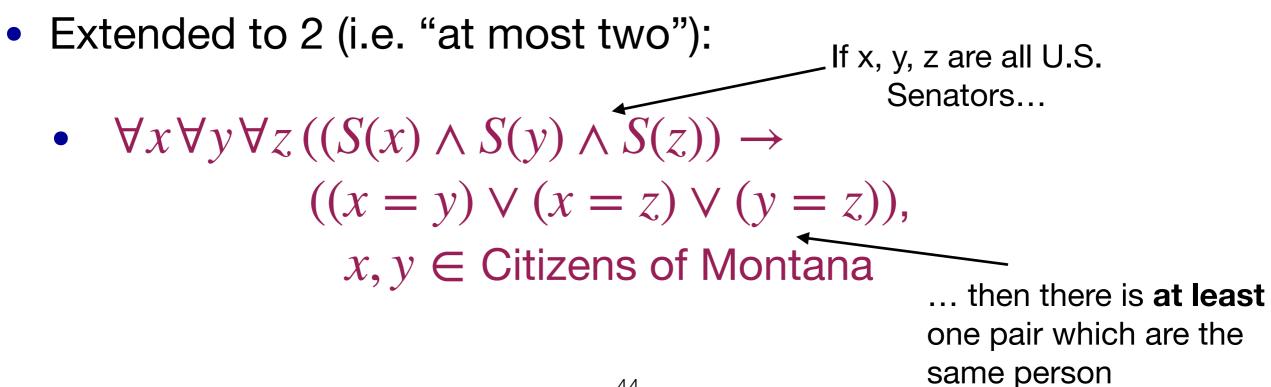
- How do we express "At least two citizens of Montana are U.S. Senators"? Let S(x) : x is a U.S. Senator, $x \in$ People
- Why doesn't this work?
 - $\exists x \exists y (S(x) \land S(y)), x, y \in \text{Citizens of Montana} (x, y could be identical)}$
- Correct version:
 - $\exists x \exists y (S(x) \land S(y) \land (x \neq y)), x, y \in \text{Citizens of Montana}$



- Consider the two halves separately:
 - 1. "At least two citizens of Montana are U.S. Senators" S(x) : x is a U.S. Senator, $x \in \text{People}$ $\exists x \exists y(S(x) \land S(y) \land (x \neq y)),$ $x, y \in \text{Citizens of Montana}$
 - 2. "At most two citizens of Montana are U.S. Senators"

Expressing "At most two"

- How do we express "At most two citizens of Montana are U.S. Senators"? Let S(x) : x is a U.S. Senator, $x \in$ People
- Start with "at most one":
 - $\forall x \forall y (S(x) \land S(y)) \rightarrow (x = y)$



- Consider the two halves separately:
 - 1. "At least two citizens of Montana are U.S. Senators" S(x) : x is a U.S. Senator, $x \in \text{People}$ $\exists x \exists y(S(x) \land S(y) \land (x \neq y)),$ $x, y \in \text{Citizens of Montana}$
 - 2. "At most two citizens of Montana are U.S. Senators"

 $\forall x \forall y \forall z ((S(x) \land S(y) \land S(z)) \rightarrow \\ (x = y \lor y = z \lor x = z)) \\ x, y, z \in \textbf{Citizens of Montana}$

• Finally, AND together

 $\exists x \exists y (S(x) \land S(y) \land (x \neq y))$

• and

$$\forall x \forall y \forall z ((S(x) \land S(y) \land S(z)) \rightarrow (x = y \lor y = z \lor x = z))$$

• Finally, AND together

 $\exists x \exists y (S(x) \land S(y) \land (x \neq y))$

and

$$\forall x \forall y \forall z ((S(x) \land S(y) \land S(z)) \rightarrow (x = y \lor y = z \lor x = z))$$

 $\exists x \exists y (S(x) \land S(y) \land (x \neq y) \land \forall z [S(z) \rightarrow (z = x \lor z = y)]),$ $x, y, z \in \textbf{Citizens of Montana}$

Why is the second half simplified?

Reminders

- Homework 2 due <u>this</u> Friday (06/19)
- Grades for Homework 1 should be posted before Friday
- Quiz 1 this Tuesday (on Logic)