Quantification

From last time:

Definition: *Proposition*

A declarative sentence that is either true (**T**) or false (**F**), but not both.

- Tucson summers never get above 100 degrees
- If the doorbell rings, then my dog will bark.
- You can take the flight if and only if you bought a ticket
- But **NOT** $x < 10$

Why do we want use variables?

- Propositional logic is not always expressive enough
- Consider: “All students like summer vacation”
- Should be able to conclude that “If Joe is a student, he likes summer vacation”.
- Similarly, “If Rachel is a student, she likes summer vacation” and so on.
- Propositional Logic does not support this!

Predicates

Definition: *Predicate (a.k.a. Propositional Function)*

A statement that includes at least one variable and will evaluate to either **true** or **false** when the variables(s) are assigned value(s).

- Example:

$$S(x) : (-10 < x) \wedge (x < 10)$$

$$E(a, b) : a \textbf{ eats } b$$

- These are not complete!

Predicates

Definition: *Domain (a.ka. Universe) of Discourse*

The collection of values from which a variable's value is drawn.

- Example:

$$S(x) : (-10 < x) \wedge (x < 10),\ \ x \in \mathbb{Z}$$

$$E(a, b) : a \textbf{ eats } b,\ a \in \textbf{People},\ b \in \textbf{Vegetables}$$

- In This Class: Domains may **NOT** hide operators
  - OK: **Vegetables**
  - Not-OK: **Raw Vegetables** **(Vegetable ∧ ¬Cooked)**

Predicates

$$S(x) : (-10 < x) \wedge (x < 10),\ \ x \in \mathbb{Z}$$

$$E(a, b) : a \textbf{ eats } b,\ a \in \textbf{People},\ b \in \textbf{Vegetables}$$

- Can evaluate predicates at specific values (making them propositions):
  - What is $E(Joe, Asparagus)$?
  - What is $S(0)$?

Combining Predicates with Logical Operators

- In $E(a, b)$: a eats b, $a \in$ people, $b \in$ vegetables, change the domain of b to "raw vegetables".
  - $E(a, b)$: a eats b, $a \in$ people, $b \in$ vegetables
  - $C(b)$: b is cooked, $b \in$ vegetables
- Combining the two:
  - $E(a, b) \wedge \neg C(b)$, $a \in$ people, $b \in$ vegetables

Quantification

- Idea: Establish truth of predicates over sets of values.
- Two common generalizations:
  - *Universal Quantification* ($\forall x P(x), x \in D$)
    - Considers all values from the domain of discourse
  - *Existential Quantification* ($\exists x\, S(x), x \in D$)
    - Considers one or more values form the domain of discourse

Note: Do not use the books non-standard $\exists ! x$ notation
("uniqueness quantifier", Rosen 8/e p.46)

Quantifications in Propositional Logic

- *Universal Quantification*
  - $\forall x P(x), x \in D \equiv P(d_0) \wedge P(d_1) \wedge P(d_2) \wedge \ldots$

- *Existential Quantification*
  - $\exists x\, S(x), x \in D \equiv S(d_0) \vee S(d_1) \vee S(d_2) \vee \ldots$

Evaluating Quantified Predicates

- *Universal Quantification:*

  Universal quantification of $P(x)$, $\forall x\, P(x)$, is the statement "$P(x)$ holds for all objects x in the domain of discourse"

  $\forall x\, P(x)$ is true only when $P(x)$ is true for **every** x in the domain, and false otherwise.

- Example 1:
  - $Q(x) : x = x^2, x \in \mathbb{R}$
  - $\forall x\, Q(x), x \in \{-1,0,1\}$?

  $Q(-1) =$ **False**, $-1 \neq (-1)^2$

- A value x for which $P(x)$ is false is a ***counterexample*** $\forall x\, P(x)$

Evaluating Quantified Predicates

- *Universal Quantification:*

  Universal quantification of $P(x)$, $\forall x\, P(x)$, is the statement "$P(x)$ holds for all objects x in the domain of discourse"

  $\forall x\, P(x)$ is true only when $P(x)$ is true for **every** x in the domain, and false otherwise.

- Example 2:
  - $P(x, y) : x + y$ is even, $x, y \in \mathbb{Z}$
  - $\forall x \forall y\ P(x, y),\ x, y \in \mathbb{Z}^{odd}$ **True! (Easy to prove)**

Evaluating Quantified Predicates

- *Existential Quantification:*

  Existential quantification of $P(x)$, $\exists x\, P(x)$, is "There exists an element x in the domain of discourse such that $P(x)$"

  $\exists x\, P(x)$ is true if **at least one** element x in the domain such that $P(x)$ is true

- Example 1:
  - $Q(x) : x = x^2, x \in \mathbb{R}$
  - $\exists x\, Q(x), x \in \{-1,0,1\}$?

$Q(0) =$ **True,**

$0 \neq 0^2$

Evaluating Quantified Predicates

- *Existential Quantification:*

  Existential quantification of $P(x)$, $\exists x\, P(x)$, is "There exists an element x in the domain of discourse such that $P(x)$"

  $\exists x\, P(x)$ is true if **at least one** element x in the domain such that $P(x)$ is true

- Example 2:
  - $P(x, y) : x + y$ is even, $x, y \in \mathbb{Z}$
  - $\exists x \exists y\ P(x, y),\ x, y \in \mathbb{Z}^{odd}$

True! Universal quantifier covers existential

Examples: Converting from English to Quantified Predicates

Example: Universal Quantification

- Consider this conversational English statement:

    Every actor in The Office is great.

- How can we express that statement in logic notation?

— — WARNING — —

Several **INCORRECT** versions follow…

Only the last version is correct!

Example: Universal Quantification

- Consider this conversational English statement:

Every actor in The Office is great.

- How can we express that statement in logic notation?

Let $P(x)$: Actor x is great, $x \in$ **People**

$$\forall x \ P(x), x \in \textbf{People}$$

Stilted English: For every person x, x is a great actor.
Conversationally: All people are great actors.

PROBLEM: This is not quite the desired meaning

IDEA: Let's focus the domain!

Example: Universal Quantification

- Attempt #2: Every actor in The Office is great.

Let $P(x)$: Actor x is great, $x \in$ **People who act in The Office**

$$\forall x \ P(x), x \in \textbf{People who act in the Office}$$

Stilted English: For every person x who acts in The Office, x is a great actor.

Conversationally: All actors in the Office are great.

PROBLEM: The domain has a hidden predicate

IDEA: Let's create a new predicate.

Example: Universal Quantification

- Attempt #3: Every actor in The Office is great.

Let $P(x)$: Actor x is great, $x \in$ **People**

Let $Q(x)$: x **was in The Office**, $x \in$ **People**

$$\forall x\ (P(x) \wedge Q(x)), x \in \textbf{People}$$

Stilted English: For every person x, x is in The Office and x is a great actor.

Conversationally: All people are in the Office and are great.

PROBLEM: If true, implies that *all people* are in The Office!

IDEA: Try a different compound predicate

Example: Universal Quantification

- Attempt #4: Every actor in The Office is great.

  Let $P(x)$: Actor x is great, $x \in$ **People**

  Let $Q(x)$: x **was in The Office**, $x \in$ **People**

  $\forall x\ (Q(x) \rightarrow P(x)), x \in$ **People**

Stilted English: For every person x, if x was in The Office, then x is a great actor.

Conversationally: Every actor in the Office is great.

PROBLEM: Isn't $P(x)$, really two predicates in one?

IDEA: Break it apart

[Why not $P(x) \rightarrow Q(x)$? That says all great actors are in the office]

Example: Universal Quantification

- Attempt #5: Every actor in The Office is great.

Let $P(x)$: x is an actor, $x \in$ **People**

Let $A(x)$: x is a great actor, $x \in$ **People**

Let $Q(x)$: x **was in The Office**, $x \in$ **People**

$$\forall x \; ((Q(x) \wedge P(x)) \rightarrow A(x)), x \in \textbf{People}$$

Stilted English: For every person x, if x is an actor and was in The Office, then x is a great actor.

Conversationally: Every actor in the Office is great.

— —SUCCESS! — —

(This is the version to learn!)

Implicit Quantification

- The "all" can be implicit in the English statement
- Example:
  - Adding an odd # to itself produces an even #

$O(x)$: x **is odd**, $x \in \mathbb{R}$

$E(x)$: x **is even**, $x \in \mathbb{R}$

$$\forall x \ (O(x) \rightarrow E(x + x)), \ x \in \mathbb{Z}$$

$$\forall x \ (O(x) \rightarrow \overline{O(x + x)}), \ x \in \mathbb{Z}$$

Note the implicit ∀ is implicit in the sentence

Example: Existential Quantification

- Consider this conversational English statement:

At least one breed of dog is cute.

- How can we express that statement in logic notation?

Let $C(x)$: x **is cute**, $x \in$ **Dog Breeds**

$\exists x \; C(x)$, $x \in$ **Dog Breeds**

English: There is at least one dog breed x such that x is cute.

Example: Existential Quantification

- Express this more specific statement in logic:

Some of the large fluffy dog breeds are cute.

Let $L(x)$: x **is large**, $x \in$ **Dog Breeds**

Let $F(x)$: x **is fluffy**, $x \in$ **Dog Breeds**

Let $C(x)$: x **is cute**, $x \in$ **Dog Breeds**

$\exists x\,(L(x) \wedge F(x) \wedge C(x))$, $x \in$ **Dog Breeds**

These alternatives don't work! Why?

$\exists x\,(L(x) \wedge F(x)) \rightarrow C(x)$, $x \in$ **Dog Breeds**

$\exists x\,(L(x) \wedge C(x)) \rightarrow F(x)$, $x \in$ **Dog Breeds**

Example: Existential Quantification

- Express this statement in logic:

Every last one of the large fluffy dog breeds are cute.

Let $L(x)$: x **is large**, $x \in$ **Dog Breeds**

Let $F(x)$: x **is fluffy**, $x \in$ **Dog Breeds**

Let $C(x)$: x **is cute**, $x \in$ **Dog Breeds**

$\forall x\, [(L(x) \wedge F(x)) \rightarrow C(x)]$, $x \in$ **Dog Breeds**

If a dog is both large and fluffy, then it is cute.

(vs. … $\wedge\, C(x)$: **All** dog breeds are large, fluffy and cute

---

Typically $\forall$ pairs with $\rightarrow$ and $\exists$ goes with $\wedge$

Nested Quantifiers

- Sometimes we may need to use multiple quantifiers
  - We can't express "Everybody loves someone" using a single quantifier.
  - Suppose predicate **loves**(x, y)**:** "Person x loves person y"

Nested Quantifiers

- **loves**(x, y)**:** "Person x loves person y"

- The four possible nestings:

Same quantifiers

- $\forall x \forall y$ **loves**(x, y)
- $\exists x \exists y$ **loves**(x, y)

Mixed quantifiers

- $\exists x \forall y$ **loves**(x, y)
- $\forall x \exists y$ **loves**(x, y)

Evaluating Nested Quantifiers of the Same Type

- Example: **loves**(x, y)**:** "Person x loves person y"
  - $\forall x \forall y$ **loves**(x, y)
    - "Everyone loves everyone".
  - $\exists x \exists y$ **loves**(x, y)
    - "There is someone who loves someone else" (or possibly themself!)

Evaluating Mixed Quantifiers

- Example: **loves**(x, y)**:** "Person x loves person y"
  - $\exists x \forall y$ **loves**(x, y)
    - "There is someone who loves everyone"
  - $\forall x \exists y$ **loves**(x, y)
    - "Everyone loves at least one person (possibly themself!)"

Evaluating Mixed Quantified

- Distinguishing $\exists x \forall y\ S(x, y)$ from $\forall i \exists k\ T(i, k)$
  - $\exists x \forall y\ S(x, y)$ - "There exists an x such that, for every y, $S(x, y)$ is true."
    - Somewhere in x's domain is an x that can be paired with any y's domain to make $S(x, y)$ true.
  - $\forall i \exists k\ T(i, k)$- "For any i there exists a k such that $T(i, k)$ is true.
    - No matter which i is selected, we can find some k to pair with the i to make $T(i, k)$ true. (Note that the k may vary with the i)

Evaluating Mixed Quantified

- Example:
  - Given $P(x, y) : x - y = 0, x, y \in \mathbb{Z}$
  - Evaluate: $\exists x \forall y \; P(x, y)$

Evaluating Mixed Quantified

- Example:
  - Given $P(x, y) : x - y = 0, x, y \in \mathbb{Z}$
  - Evaluate: $\exists x \forall y \ P(x, y)$ = **False**
    - (No such magical integer x exists!)

x	y	$P(x, y)$
0	$\vdots$	$\vdots$
0		
0		
0	-1	F
0	0	T
0	1	F
0	$\vdots$	$\vdots$
0		
0		

Evaluating Mixed Quantified

- Example:
  - Given $P(x, y) : x - y = 0, x, y \in \mathbb{Z}$
  - Evaluate: $\exists x \forall y \ P(x, y)$ = **False**
  - Evaluate: $\forall x \exists y \ P(x, y)$ = **True**
    - (No matter the x, there's an integer y ($y = x$) that makes $P(x, y)$ true.)

x	y	$P(x, y)$
-3	-3	T
-2	-2	T
-1	-1	T
0	0	T
1	1	T
2	2	T
3	3	T

Negation of Quantified Expressions

- Remember De Morgan's Laws for Propositions? Well…

Definition: *Generalized De Morgan's Laws*

The Generalized De Morgan's Laws are the pair of equivalences:

$$\neg \forall x P(x) \equiv \exists \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall \neg P(x)$$

Demonstration: $\neg\forall x\, P(x) \equiv \exists x \neg P(x)$

- **Reminder:**

$$\forall x\, S(x), x \in D \equiv S(d_1) \wedge S(d_2) \wedge S(d_3)\ldots$$

$$\exists x\, S(x), x \in D \equiv S(d_1) \vee S(d_2) \vee S(d_3)\ldots$$

Demonstration: $\neg \forall x\, P(x) \equiv \exists x \neg P(x)$

- Let $S(x)$: x is cute, $x \in D$. Let $D =$ {all dog breeds}

$$\forall x\, S(x), x \in D \equiv S(d_1) \wedge S(d_2) \wedge S(d_3) \ldots$$

$$\neg \forall x\, S(x), x \in D \equiv \neg (S(d_1) \wedge S(d_2) \wedge S(d_3) \ldots)$$

$$\equiv \neg S(d_1) \vee \neg S(d_2) \vee \neg S(d_3) \ldots$$

$$\equiv \exists x\ \neg S(x),\ x \in D$$

$$\exists x\, S(x), x \in D \equiv S(d_1) \vee S(d_2) \vee S(d_3) \ldots$$

Negation of Nested Quantifiers

- Apply De Morgan's Laws from left to right
- Example:

$$\neg(\exists x \forall y(G(x) \vee \neg H(y)))$$

$$\equiv \forall x \neg(\forall y(G(x) \vee \neg H(y)))$$ **[General DeMorgan]**

$$\equiv \forall x \exists y \neg(G(x) \vee \neg H(y))$$ **[General DeMorgan]**

$$\equiv \forall x \exists y(\neg G(x) \wedge H(y))$$ **[DeMorgan]**

Expressing "Exactly one..." Statements

- Consider the conversational (& correct!) English statement

  Only one citizen of Montana is a member of the U.S. House of Representatives

- And consider this awkward but useful rewording:

  A member of the US House of Representatives exists in the set of citizens of Montana, and, if anyone in Montana is a member of the House, that person is the representative

Expressing "Exactly one…" Statements

- That rewording is useful because it can be directly expressed logically:

  $R(x)$: x is a member of the US House of Representatives, $x \in$ **People**

  $\exists x(R(x) \wedge \forall y[R(y) \rightarrow (y = x)]), x, y \in$ **Citizens of Montana**

This domain should be simplified, but using it makes the logic easier to read (for now)

Expressing "Exactly one…" Statements

- That rewording is useful because it can be directly expressed logically:

  $R(x)$: x is a member of the US House of Representatives, $x \in$ **People**

  $\exists x(R(x) \wedge \forall y[R(y) \rightarrow (y = x)]), x, y \in$ **Citizens of Montana**

- Interpretation: (At least one) ∧ (No more than one)
- ∴ Impossible for there to be two representatives!

Expression "Exactly two…" Statement

- Key observation:

  Exactly 2 ≡ At least 2 ∧ At most 2

  $$(n = 2) \equiv (n \geq 2) \wedge (n \leq 2)$$

- Awkward English:

  At least two citizens of Montana are U.S. Senators, and at most two citizens of Montana are U.S. Senators.

- Better:

  Exactly two citizens of Montana are U.S. Citizens

Expression “Exactly two…” Statement

- Consider the two halves separately:

1. “At least two citizens of Montana are U.S. Senators”

2. “At most two citizens of Montana are U.S. Senators”

Expressing "At least two"

- How do we express "At least two citizens of Montana are U.S. Senators"? Let $S(x)$: x is a U.S. Senator, $x \in$ People

- Why doesn't this work?
  - $\exists x \exists y\, (S(x) \wedge S(y))$, $x, y \in$ Citizens of Montana **(x, y could be identical)**

- Correct version:
  - $\exists x \exists y\, (S(x) \wedge S(y) \wedge (x \neq y))$, $x, y \in$ Citizens of Montana

x and y are both U.S. Senators

and

they are not the same person

Expression "Exactly two…" Statement

- Consider the two halves separately:

1. "At least two citizens of Montana are U.S. Senators"

   $S(x)$: x is a U.S. Senator, $x \in$ **People**

   $\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y))$,

   $x, y \in$ **Citizens of Montana**

2. "At most two citizens of Montana are U.S. Senators"

Expressing "At most two"

- How do we express "At most two citizens of Montana are U.S. Senators"? Let $S(x)$: x is a U.S. Senator, $x \in$ People

- Start with "at most one":

  - $\forall x \forall y \, (S(x) \wedge S(y)) \rightarrow (x = y)$

- Extended to 2 (i.e. "at most two"):

  - $\forall x \forall y \forall z \, ((S(x) \wedge S(y) \wedge S(z)) \rightarrow$
    $((x = y) \vee (x = z) \vee (y = z)),$
    $x, y \in$ Citizens of Montana

If x, y, z are all U.S. Senators…

… then there is **at least** one pair which are the same person

Expression "Exactly two…" Statement

- Consider the two halves separately:

1. "At least two citizens of Montana are U.S. Senators"

   $S(x)$: x is a U.S. Senator, $x \in$ **People**

   $\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y)),$

   $x, y \in$ **Citizens of Montana**

2. "At most two citizens of Montana are U.S. Senators"

   $\forall x \forall y \forall z ((S(x) \wedge S(y) \wedge S(z)) \rightarrow$

   $(x = y \vee y = z \vee x = z))$

   $x, y, z \in$ **Citizens of Montana**

Expression "Exactly two…" Statement

- Finally, **AND** together

$$\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y))$$

- and

$$\forall x \forall y \forall z ((S(x) \wedge S(y) \wedge S(z)) \rightarrow (x = y \vee y = z \vee x = z))$$

Expression "Exactly two..." Statement

- Finally, **AND** together

$$\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y))$$

- and

$$\forall x \forall y \forall z ((S(x) \wedge S(y) \wedge S(z)) \rightarrow$$
$$(x = y \vee y = z \vee x = z))$$

$$\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y) \wedge$$
$$\forall z [S(z) \rightarrow (z = x \vee z = y)]),$$
$x, y, z \in$ **Citizens of Montana**

Why is the second half simplified?

Reminders

- Homework 2 due **this** Friday (06/19)
- Grades for Homework 1 should be posted before Friday
- Quiz 1 **this** Tuesday (on Logic)