Relations

Section 9.1, 9.3, 9.5, 9.6

Background

- Having collections of data: Good
- Knowing the connections between collections: Better!
- **Example**:
    - Students - Courses
    - Businesses - Email Adresses
    - Dogs - Trees

Relations

Definition: *(Binary) Relation*

A binary relation from set X to Y is a subset of the Cartesian Product of X (the domain) and Y (the codomain).

NOTE: a relation "on set W" $\equiv$ "from set W to set W".

Example:

$A = \{$**Leslie Knope, Jim Halpert, Michael Scott, Ann Perkins, Ben Wyatt**$\}$

$B = \{$**Parks and Rec, The Office**$\}$

$R = \{($**Leslie Knope, Parks and Rec**$), ($**JH, O**$), ($**MS, O**$), ($**AP, P&R**$), ($**BW, P&R**$)\}$

Relations

Definition: *Related*

If $(x, y) \in R$, x is **related to** y ($x \, R \, y$)

Example:

(**Leslie Knope, Parks and Rec**) $\in R$ (**Leslie Knope** R **Parks and Rec**)

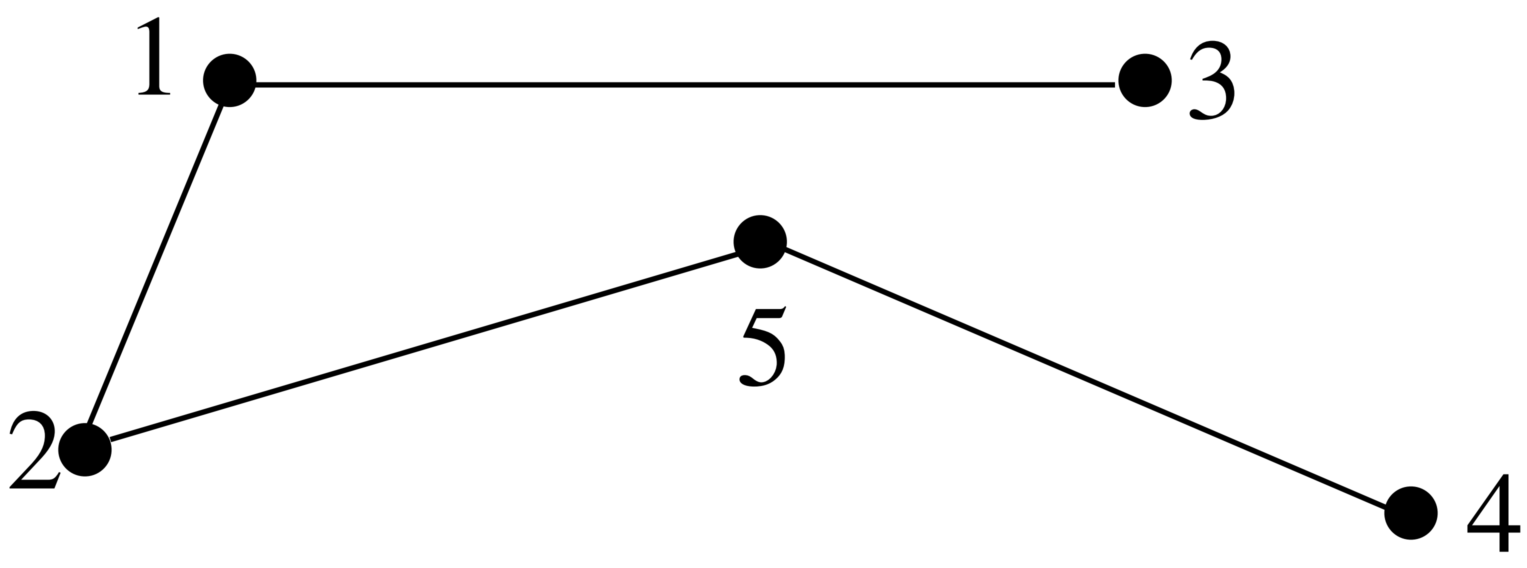
(**Jim Halpert, Parks and Rec**) $\notin R$ (**Jim Halpert** $\not R$ **Parks and Rec**)

Let $H = \{1,2,3,4,5,6\}$ and let R be a relation on H such that $x \; R \; y$ when $x \,\%\, y = 0$, $x \neq y$

$R = \{(2,1), (3,1), (4,1), (5,1), (6,1), (4,2), (6,2), (6,3)\}$

Graph Representations of Relations

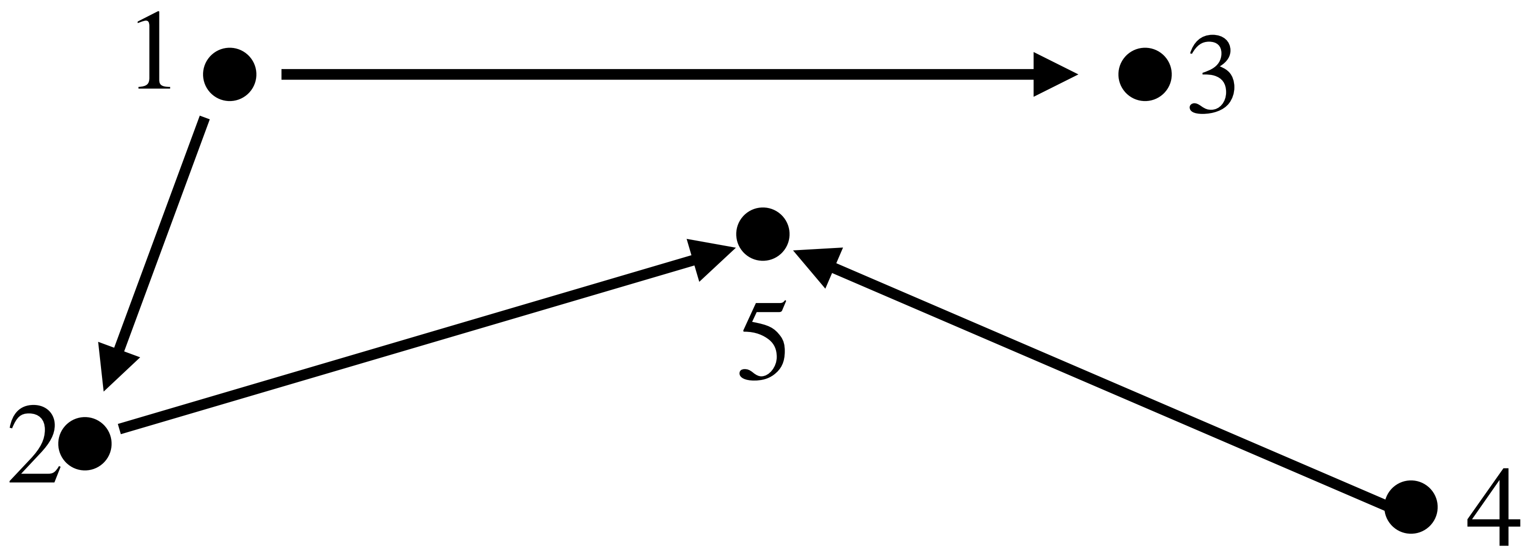
- **Graphs**:
  - A set V of vertices (nodes) and a set E of pairs of vertices that represent an edge between those two vertices



$$V = \{1,2,3,4,5\}$$
$$E = \{(1,3), (1,2), (2,5), (4,5)\}$$

Graph Representations of Relations

- **Directed Graphs (Digraph)**:
  - A set V of vertices (nodes) and a set E of pairs of vertices that represent an edge between those two vertices
  - In edge (a,b), a is the initial vertex and b is the terminal vertex



$$V = \{1,2,3,4,5\}$$
$$E = \{(1,3), (1,2), (2,5), (4,5)\}$$

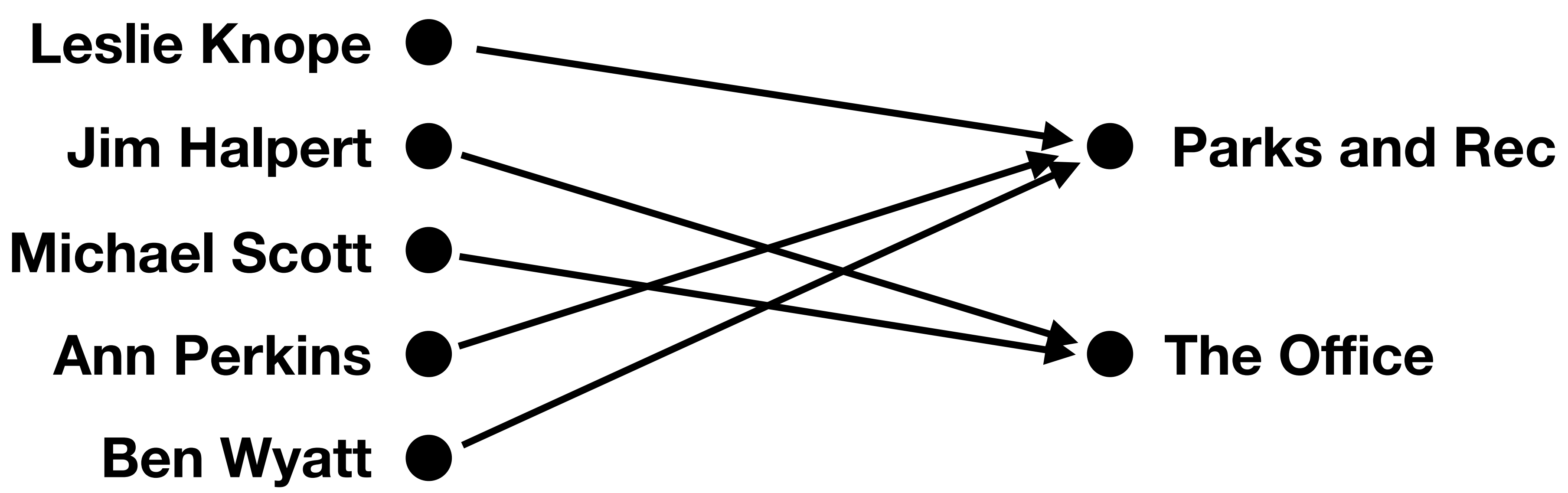
Graph Representations of Relations

- **Example:**

$$A = \{\textbf{Leslie Knope, Jim Halpert, Michael Scott,}$$
$$\textbf{Ann Perkins, Ben Wyatt}\}$$

$$B = \{\textbf{Parks and Rec, The Office}\}$$

$$R = \{(\textbf{Leslie Knope, Parks and Rec}), (\textbf{JH, O}),$$
$$(\textbf{MS, O}), (\textbf{AP, P\&R}), (\textbf{BW, P\&R})\}$$

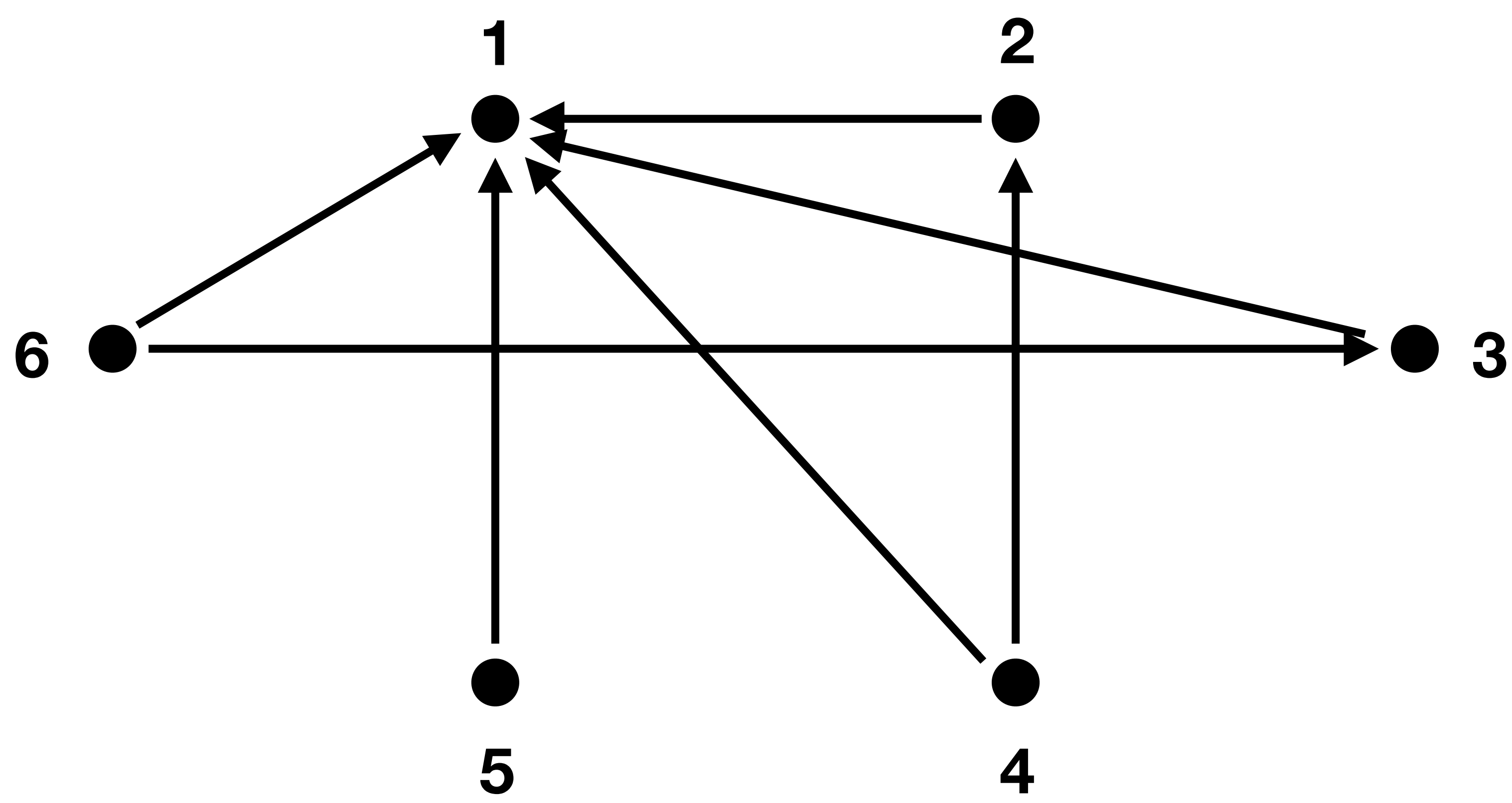


Graph Representations of Relations

- **Example:** $x \% y = 0,\ x \neq y$

Recall: $H = \{1,2,3,4,5,6\}$

$R = \{(2,1), (3,1), (4,1), (5,1), (6,1), (4,2), (6,2), (6,3)\}$



Note: Vertices with just one outgoing edge are prime

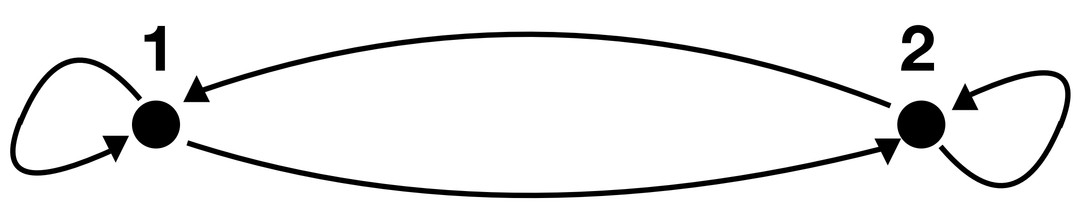
Properties of Relations

Definition: *Reflexivity*

A relation R on set A is reflexive when $(a, a) \in R$, $\forall a \in A$

Example:

$$\{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$$



(A directed edge whose source is also the destination is a self-loop)

Properties of Relations

Definition: *Symmetry*

A relation R on set A is symmetric if $(a, b) \in R$ whenever $(b, a) \in R$, for $a, b \in A$

(All non-self-loop edges are 'back-and-forth')

Example:

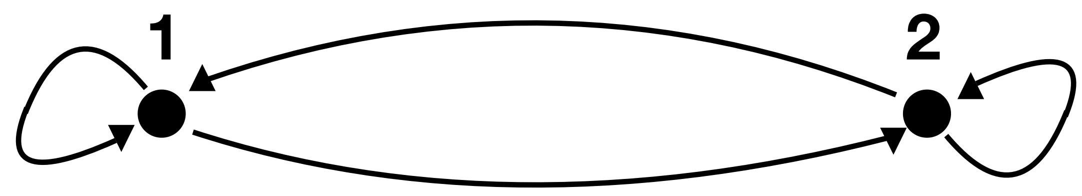
$R = \{(1,1), (1,2), (2,1), (2,2)\}$ **on** $A = \{1,2\}$ **is symmetric**

$R = \{(a, c), (a, d), (c, a), (b, c)\}$ **on** $A = \{a, b, c, d\}$ **is not symmetric (**(d, a) **and** (c, b) **are missing.)**

Properties of Relations

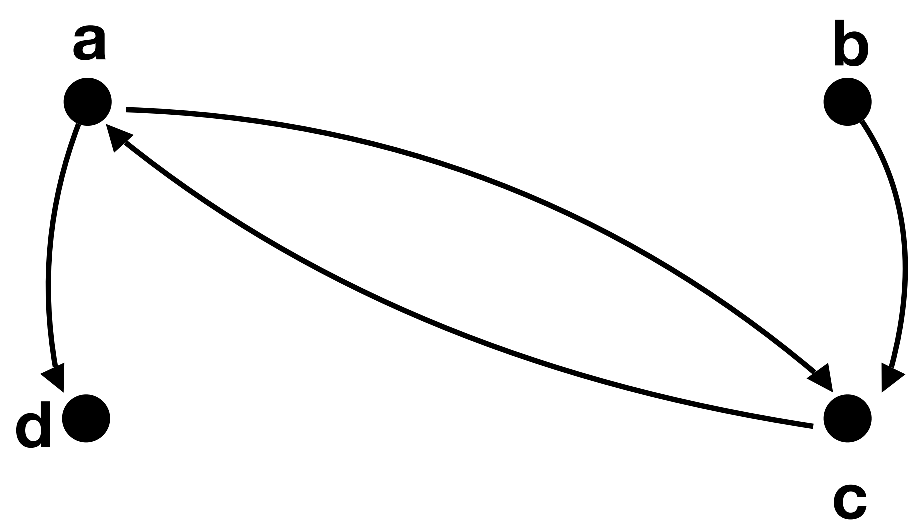
Example: Graph Representation & Symmetry

$R = \{(1,1), (1,2), (2,1), (2,2)\}$**:**



(Excepting self-loops, just have back-and-forth arrows)

$R = \{(a,c), (a,d), (c,a), (b,c)\}$**:**



(Easy to see that (d,a) **and** (c,b) **are missing**

Properties of Relations

Definition: *Antisymmetry*

A relation R on set A is antisymmetric if $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$, $\forall x, y \in A$.

(No non-self-loop edges are 'back-and-forth')

Example:

$\{(3,4)\}$ **on** $\{3,4\}$ **is antisymmetric (**$(4,3)$ **is not present)**

$\{(1,1), (3,1), (1,3)\}$ **on** $\{1,3\}$ **is not antisymmetric**

$\{(a, b), (a, d), (c, a), (b, c)\}$ **is not**

(Thus, relations may be neither symmetric nor antisymmetric.)

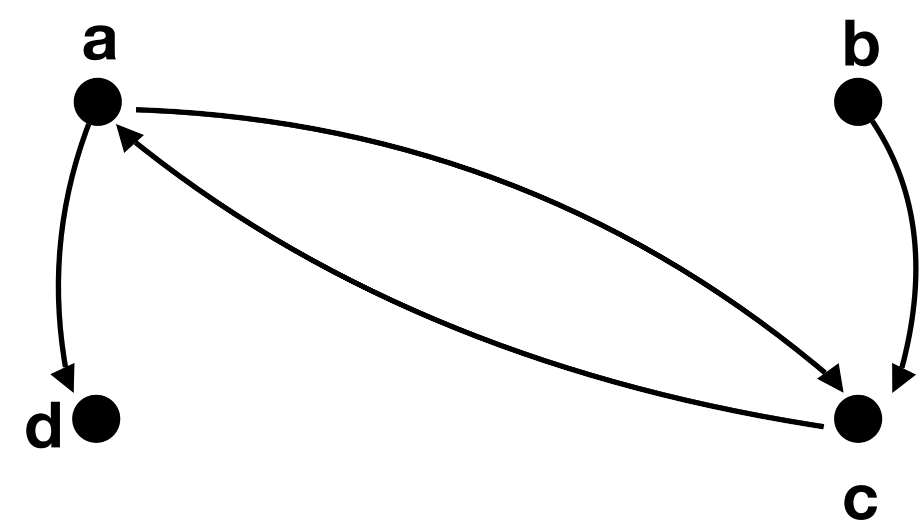
Properties of Relations

Example: Graph Representation & Antisymmetry

$R = \{(a,c),(a,d),(c,a),(b,c)\}$ **as a digraph**

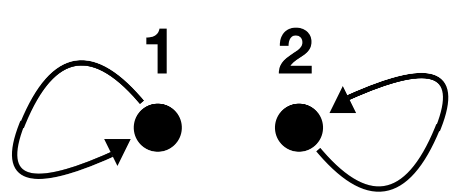
(The offending 'double edge'

Is easy to see)



$R = \{(1,1),(2,2)\}$ **on** $\{1,2\}$

(both symmetric & antisymmetric) as a digraph:



Properties of Relations

Definition: *Transitivity*

A relation R on set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, where $a, b, c \in A$

Example:

$R = \{(x, y), (x, z), (y, z), (z, x), (z, y)\}$ **on** $\{x, y, z\}$

(x, y) & $(y, z) \Rightarrow (x, z)$, **which is** $\in R$

(y, z) & $(z, y) \Rightarrow (y, y)$ **which is not in** R

$\therefore R$ **is not transitive**

Properties of Relations

Example:

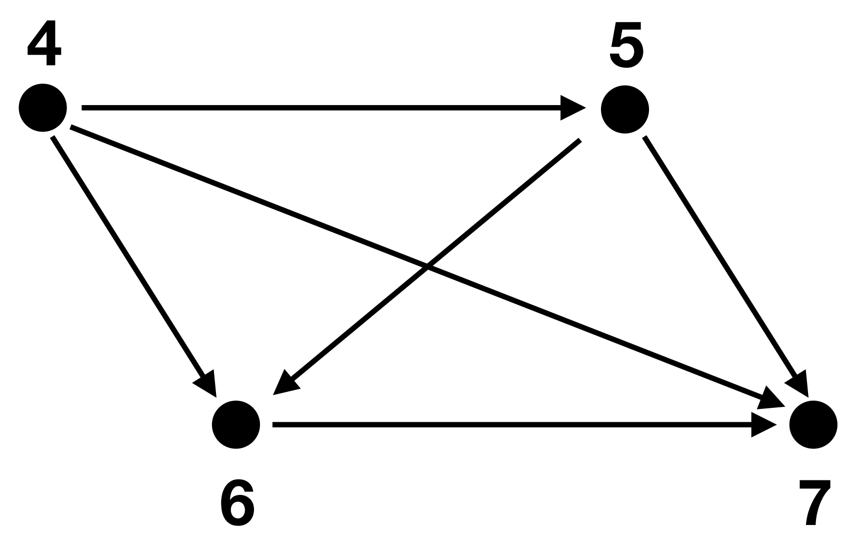
$S = \{(4,5), (4,6), (4,7), (5,6), (5,7), (6,7)\}$ **on** $\{4,5,6,7\}$

$(4,5)\ \&\ (5,6) \Rightarrow (4,6)$

$(4,5)\ \&\ (5,7) \Rightarrow (4,7)$

$(4,6)\ \&\ (6,7) \Rightarrow (4,7)$

$(5,6)\ \&\ (6,7) \Rightarrow (5,7)$



$\therefore S$ **is transitive**

(Note: Digraphs don't really help see transitivity)

Relational Composition Examples

- Three examples of creating relations from relations

- **Example #1:** Set operators

Recall: A relation is a set of ordered pairs

Let $A = \{1,2,3\}$

$R = \{(1,2), (1,3)\}$ **on** A

$S = \{(1,1), (2,3)\}$ **on** A

$R \cup S = \{(1,2), (1,3), (1,1), (2,3)\}$ **on** A

is also a relation on A

Relational Composition Examples

- **Example #2:** Swapping content of ordered pairs $(1,2) \Rightarrow (2,1)$

Definition: *Inverse*

The inverse of a relation R, denoted R^{-1}, contains all of the ordered pairs of R with their components exchanged

That is: $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Relational Composition Examples

- **Example #3:** Composites **Remember:** $f \circ g = f(g(x))$

Definition: *Composite*

Let G be a relation from A to B, and F be a relation from B to C. The composite of F and G, $F \circ G$, is the relation of ordered pairs (a, c), $a \in A$, $c \in C$, such that $(a, b) \in G$ and $(b, c) \in F$, where $b \in B$

Example:

Let $X = \{(1,a), (2,b), (3,c)\}$

$Y = \{(1,2), (2,3), (1,3), (2,4)\}$

$X \circ Y = \{(1,b), (2,c), (1,c)\}$

Relational Composition Examples

- **Example #3:** Composites (cont)

Example:

Let $C = \{(\alpha, -4), (\alpha, -2), (\beta, -6), (\gamma, -4)\}$

$D = \{(q, \beta), (x, \alpha), (x, \gamma)\}$

$C \circ D = \{(q, -6), (x, -4), (x, -2)\}$

(Note that $(x, -4)$ is not repeated; this is a set)

Definition: *Complement*

The complement of a relation R, denoted $\overline{R}$, is $\{(a, b) \mid (a, b) \notin R\}$

Matrix Representation of Relations

- We assume that relations are on just one set
- The 0-1 matrix representation of relation R on set A is $|A| \times |A|$, with both dimensions labeled identically. When $(a, b) \in R$, then `matrix[a][b]=1`. Else, `matrix[a][b]=0`

Example:

$$R = \{(a,c),(a,d),(c,a),(b,c)\} \text{ on } \{a,b,c,d\}$$

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Matrix Representation of Relations

- **Observation #1:** Detecting Reflexivity
  - A relation is reflexive when its corresponding matrix representation has no 0's along the main diagonal

Example:

$R = \{(1,1),(1,2),(2,2)\}$ on $\{1,2\}$

$$\begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \end{bmatrix}$$

The main diagonal is all 1's; $\therefore R$ is reflexive

Matrix Representation of Relations

- **Observation #2:** Detecting Symmetry
  - Let matrix M represent R. R is symmetric when $m_{ij} = 1$ iff $m_{ji} = 1$ is true

Example:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

When the relation is symmetric, the matrix is symmetric

Matrix Representation of Relations

- **Observation #3:** Detecting Transitivity
  - Let matrix M represent R. R is transitive when the non-zero elements of M^2 (or of $M^{([2])}$) are also non-zero in M

Example:

Is $\{(1,1),(2,2),(2,3),(3,2)\}$ on $\{1,2,3\}$ transitive?

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 1\\0 & 1 & 1\end{bmatrix}$$

So, no, the relation is not transitive

Equivalence Relations

Definition: *Equivalence Relation*

A relation on set A is an equivalence relation if it is *reflexive, symmetric,* and *transitive.*

Example:

$A = \{2,3,4\}$. Let Q be a relation on A such that $a = b$, $\forall a, b \in A$

Thus $Q = \{(2,2), (3,3), (4,4)\}$

Reflexive? Symmetric? Transitive?

Equivalence Relations

Example:

$B = \{-2, -1, 0, 1, 2\}$. Let R be a relation on B such that $|a| = |b|, \forall a, b \in B$

Thus $R = \{(0,0), (1,1), (1,-1), (-1,1), (-1,-1),$
$(2,2), (2,-2), (-2,2), (-2,-2)\}$

This is also an equivalence relation.

Equivalence Relations

So ... why are these called *equivalence* relations?

Recall:

$$R = \{(0,0),(1,1),(1,-1),(-1,1),(-1,-1),$$
$$(2,2),(2,-2),(-2,2),(-2,-2)\}$$

Note the "clusters" of 0's, 1's, and 2's. This gives a partition of the base set B :

$$\{\{0\},\{-1,1\},\{-2,2\}\}$$

Equivalence Relations

Definition: *Equivalence Class*

The equivalence relation R on set B, and an element $b \in B$, is $\{c \,|\, c \in B \wedge (b, c) \in R\}$ and is denoted $[b]$. (That is, the set of everything paired with & on the right side of b in R

Example: **(From previous slide)**

$$[1] \Rightarrow \{c \,|\, c \in B \wedge (1,c) \in R\}$$

R contains $(1,1)$ and $(1,-1)$, so $[1] = \{1,-1\}$.

Partial Orders

- Consider scheduling the construction of a house.
- Example: foundation, then walls, then paint
  - But what about bathroom tile and the kitchen sink?

Definition: *Reflexive (a.k.a. Weak) Partial Order*

A relation R on set A is a (reflexive/weak) partial order if it is *reflexive, antisymmetric,* and *transitive.*

Notation: $x \preceq y$ **means** $(x, y) \in R$
when R **is a partial order**

Partial Orders

Example:

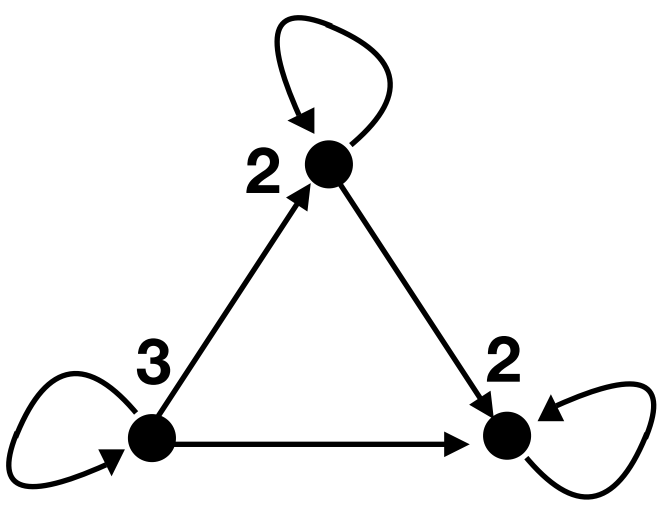
$S = \{(1,1), (1,2), (2,2), (3,1), (3,2), (3,3)\}$ **on** $\{1,2,3\}$

Is S reflexive?

Antisymmetric?

Transitive?

$\therefore S$ is a week partial order



$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

(Note: A partially-ordered set is called a *poset*)

Partial Orders

Definition: *Irreflexivity (of Relations)*

A relation R on set A is irreflexive if $\forall a \in A, (a, a) \notin R$
(Note: Not the same as "not reflexive")

Definition: *Irreflexive (a.k.a. Strict) Partial Order*

A relation R on set A is a irreflexive partial order if it is *irreflexive, antisymmetric,* and *transitive.*

(Thus no self-loops allowed.)

Total Orders

Definition: *Comparable*

Let R be a weak partial order on set A. a and b are compatible if $a, b \in A$and either $a \preceq b$ or $b \preceq a$. (That is, $(a, b) \in R$ or $(b, a) \in R$

Definition: *Total Order*

A weak-partial-ordered relation R on a set A is a total order if every pair of elements $a, b \in A$ are comparable. (Or: A relation R on A is a total order if R is antisymmetric, transitive, and comparable.)

Total Orders

Example:

$S = \{(1,1), (1,2), (2,2), (3,1), (3,2), (3,3)\}$ on $\{1,2,3\}$

It is a partial order and the paris $(1,2)$, $(3,1)$ and $(3,2)$ show that all elements are comparable. $\therefore$ Total Order!

Let $T = \{(1,1), (1,3), (2,2), (2,3), (3,3)\}$ on $\{1,2,3\}$

Reflexive? Antisym? Transitive? $\therefore$ Partial Order

But: 1 and 2 are not comparable; this is <u>not</u> a total order.