Sets

Set Concepts Covered in the Math Review

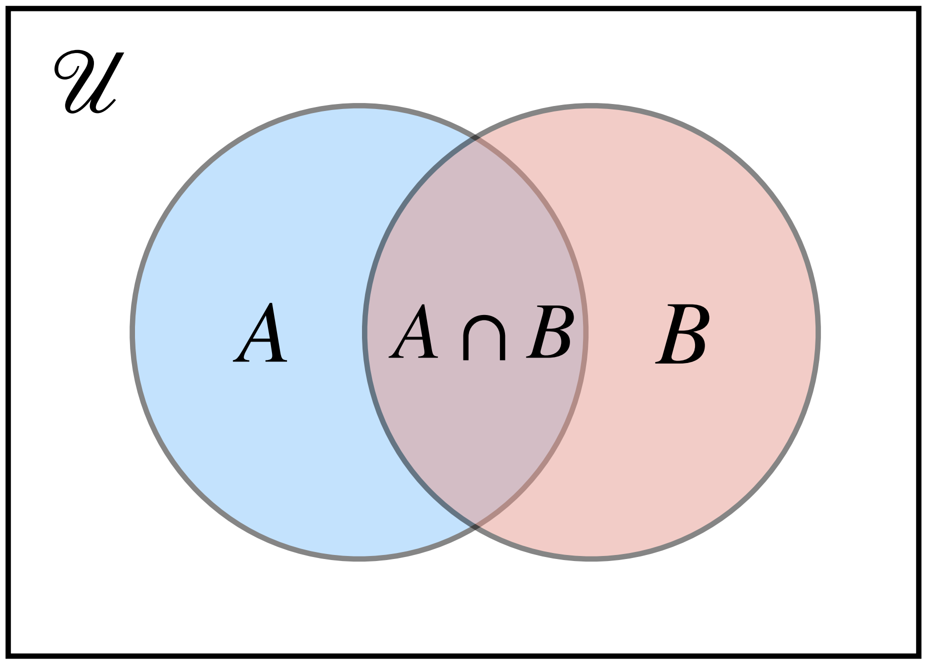
- Properties of Sets
- Set notation
- Operators
- Venn diagrams

Properties of Sets

- *Sets* are collections of unordered, distinct objects (no duplicates)
- Objects in a set are called *members* (or *elements*) of that set
- If x is a member of S, we write $x \in S$
- The number of elements in a set is called its *cardinality* written
- Infinite sets are often written using *set builder* notation

$$S = \{x \,|\, x \text{ has property } p\}$$

Venn diagrams



Why are We Studying Sets?

- Sets are foundational in many areas of Computer Science:
  - E.g.
    - Relational Model of DBMS’s
      - Based on Set theory
    - “Hard" Problems in CS
      - E.g. Set covering (what is the smallest number of special forces commandos that can be selected such that the mission team has at least one person with each necessary skill?)

Subsets & Supersets

Definition: Subset

Set A is a subset of set B ($A \subseteq B$) if every member of A can be found in B.

In other words, $A \subseteq B \equiv \forall z\,(Z \in A \to z \in B),\ z \in \mathscr{U}$

Definition: Proper Subset

Set A is a proper subset of set B ($A \subset B$) if $A \subseteq B$ and $A \neq B$. In other words, $A \subset B \equiv \forall z\,(Z \in A \to z \in B)$

$\land\ \exists w(w \notin A \land w \in B),\ z, w, \in \mathscr{U}$

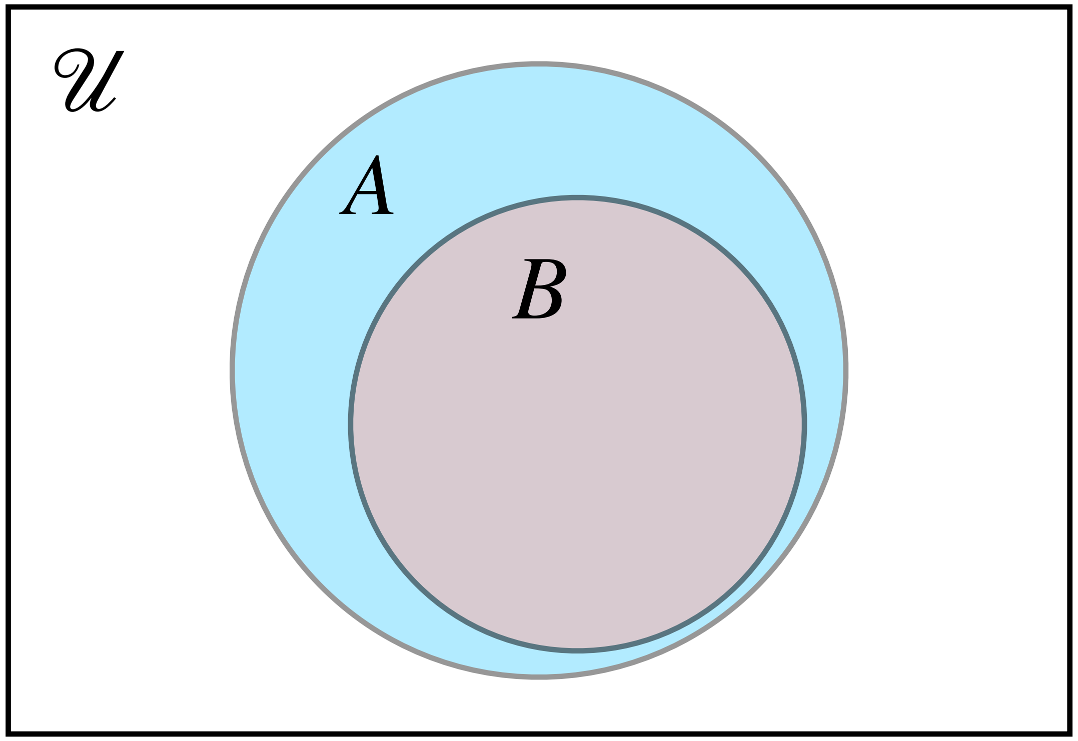
Definition: Superset

If $A \subseteq B$, then B is called a superset of A, written $B \supseteq A$

Subsets & Supersets

In Venn Diagrams:

$B \subset A$



Example: Let $G = \{1,3,4\}$ and $H = \{1,2,3,4,5\}$

Is $G \subseteq H$? **Yes**

Is $G \subset H$? **Yes**

Is $H \subseteq G$? **No**

Set Equality

Definition: *Set Equality*

Sets A and B are equal ($A = B$) iff $A \subseteq B$ and $B \subseteq A$.

Example:

Let $J = \{a, b, c, d\}$ and $K = \{b, d, c, a\}$

Is $J \subseteq K$? **Yes**

Is $J \subset K$? **No**

Is $K \subseteq J$? **Yes**

Is $K \subset J$? **No**

Does $J = K$? **Yes**

Power Sets

Definition: *Power Set*

The power set of set A, written $\mathscr{P}(A)$, is the set of all of A's subsets, including the empty set.

Example:

Let $A = \{\alpha, \beta \gamma\}$

$$\mathscr{P}(A) = \{\varnothing, \{\alpha\}, \{\beta\}, \{\gamma\},$$

$$\{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\},$$

$$\{\alpha, \beta, \gamma\}\}.$$

Note: $|\mathscr{P}(X)| = 2^{|X|}$

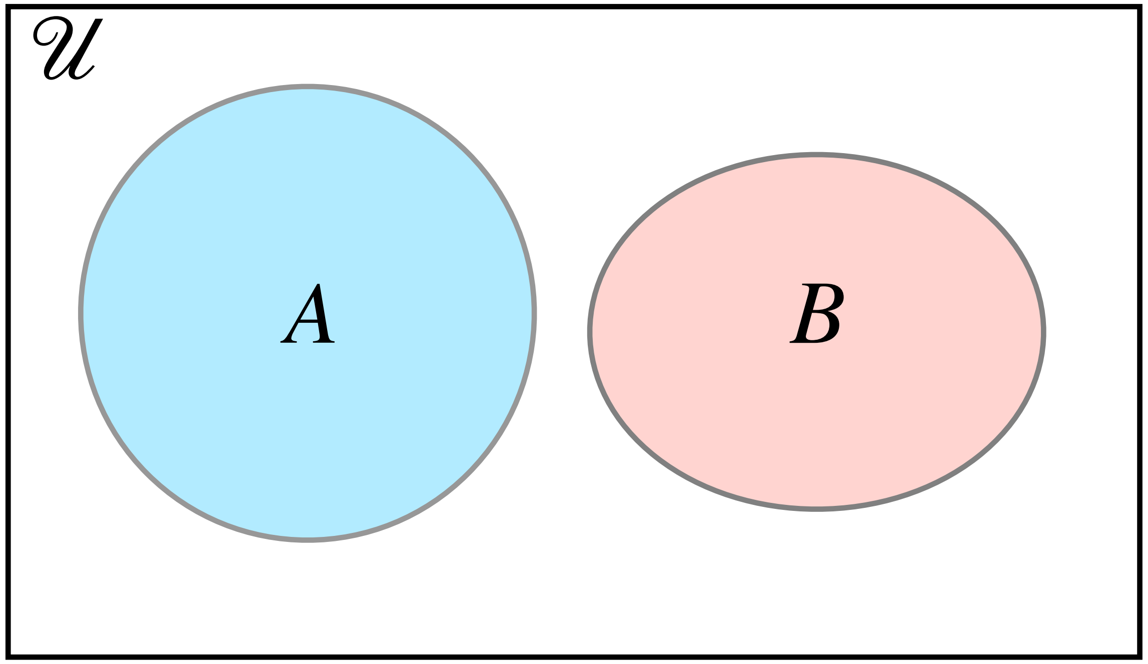
Genearlized Forms of ∪ and ∩

- Remember summation and product notation? E.g.
  - $\sum_{n=1}^{9}(2n+1)$
- Similar notation is used to generalize the union and intersection operators.
- Assuming that $A_1 \ldots A_m$ and $B_1 \ldots B_m$ are sets, then:
  - $\bigcup_{i=1}^{m} A_i = A_i \cup A_2 \cup \ldots \cup A_m$
  - $\bigcap_{i=1}^{m} B_i = B_i \cap B_2 \cap \ldots \cap B_m$

Two More Set Properties

Definition: *Disjoint*

Two sets are disjoint if their intersection is the empty set. I.e. A and B are disjoint when $A \cap B = \varnothing$



Two More Set Properties

Definition: *Disjoint*

Two sets are disjoint if their intersection is the empty set. I.e. A and B are disjoint when $A \cap B = \varnothing$

Definition: *Partition*

A separation of members of a set into disjoint subsets.

Example:

Let $C = \{a, e, i, o, u\}$ and $D = \{g, j, p, q, y\}$.

$C \cap D = \varnothing$, thus C and D are disjoint

A partition of C : $\{\{a, e\}, \{i\}, \{o, u\}\}$

Examples of Set Identities

Associativity	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$
Commutativity	$A \cap B = B \cap A$ $A \cup B = B \cup A$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Note: As with logical identities, you do not need to memorize set identities

Expressing Set Operations in Logic

- We've seen the first two already:

$$A \subseteq B \equiv \forall z\,(Z \in A \rightarrow z \in B),\ z \in \mathscr{U}$$

$$A \subset B \equiv \forall z\,(Z \in A \rightarrow z \in B) \wedge \exists w(w \notin A \wedge w \in B),\ z, w, \in \mathscr{U}$$

- For those that return sets, Set Builder notation is a good choice

$$\begin{aligned} X \cup Y &\equiv \{z | z \in X \vee z \in Y\} \\ X \cap Y &\equiv \{z | z \in X \wedge z \in Y\} \\ X - Y &\equiv \{z | z \in X \wedge z \notin Y\} \\ \overline{X} &\equiv \{z | z \notin X\} \end{aligned}$$

Proving Set Identities

- To prove that set expressions S and T are equal, we can:
    1. Prove that $S \subseteq T$ and $T \subseteq S$, or
    2. Convert the equality to logic to prove it, and convert back

Example:

To Prove $S \cup \mathscr{U} = \mathscr{U}$ (Law of Domination), either:

1. Prove both $S \cup \mathscr{U} \subseteq \mathscr{U}$ and $\mathscr{U} \subseteq S \cup \mathscr{U}$, or
2. Express with set builder notation and logic operators, prove, and convert back to set operators

Proving Set Identities

Conjecture: $S \cup \mathscr{U} = \mathscr{U}$

Proof (direct): We will show $S \cup \mathscr{U} \subseteq \mathscr{U}$ and $\mathscr{U} \subseteq S \cup \mathscr{U}$

Case 1: Demonstrate $S \cup \mathscr{U} \subseteq \mathscr{U}$

$$
\begin{aligned}
S \cup \mathcal{U} \subseteq \mathcal{U} \quad & \equiv \quad \forall z \;\; z \in (S \cup \mathcal{U}) \rightarrow z \in \mathcal{U} && \textbf{[Def of } \subseteq \textbf{]} \\
& \equiv \quad \forall z \;\; z \in (S \cup \mathcal{U}) \rightarrow \mathrm{T} && \textbf{[Def of } \mathscr{U} \textbf{]} \\
& \equiv \quad \forall z \;\; \neg z \in (S \cup \mathcal{U}) \vee \mathrm{T} && \textbf{[Law of Imp.]} \\
& \equiv \quad \forall z \; \mathrm{T} && \textbf{[Domination]} \\
& \equiv \quad \mathrm{T} && \textbf{[Tautology]}
\end{aligned}
$$

(Continued …)

Proving Set Identities

Case 2: Demonstrate $\mathscr{U} \subseteq S \cup \mathscr{U}$

$$
\begin{aligned}
\mathcal{U} \subseteq S \cup \mathcal{U} \quad &\equiv \quad \forall z \;\; z \in \mathcal{U} \rightarrow z \in S \cup \mathcal{U} && \text{[Def of } \subseteq\text{]} \\
&\equiv \quad \forall z \;\; \mathrm{T} \rightarrow z \in (S \cup \mathcal{U}) && \text{[Def of } \mathscr{U}\text{]} \\
&\equiv \quad \forall z \;\; \mathrm{T} \rightarrow (z \in S \vee z \in \mathcal{U}) && \text{[Def of } \cup\text{]} \\
&\equiv \quad \forall z \;\; \mathrm{T} \rightarrow (z \in S \vee \mathrm{T}\ ) && \text{[Def of } \mathscr{U}\text{]} \\
&\equiv \quad \forall z \;\; \mathrm{T} \rightarrow \mathrm{T} && \text{[Domination]} \\
&\equiv \quad \forall z \;\; \mathrm{T} && \text{[Def of } \rightarrow\text{]} \\
&\equiv \quad \mathrm{T} && \text{[Tautology]}
\end{aligned}
$$

Therefore, $S \cup \mathscr{U} = \mathscr{U}$

Note: Can't move from … $\rightarrow z \in S \cup \mathscr{U}$ **to**

… $\rightarrow z \in \mathscr{U}$ **because that's applying the conjecture.**

Proving Set Identities

Conjecture: $S \cup \mathscr{U} = \mathscr{U}$

Proof (direct): We will show using set builder notation

$$
\begin{aligned}
S \cup \mathcal{U} &= \{z | z \in S \vee z \in \mathcal{U}\} && \text{[Def of } \cup\text{]} \\
&= \{z | z \in S \vee \mathrm{T}\} && \text{[Def of } \mathscr{U}\text{]} \\
&= \{z | \mathrm{T}\} && \text{[Domination]} \\
&= \mathcal{U} && \text{[Def of } \mathscr{U}\text{]}
\end{aligned}
$$

Therefore, $S \cup \mathscr{U} = \mathscr{U}$

Proving Set Identities

Conjecture: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof (direct): Using set notation

$$
\begin{array}{rcll}
\overline{A \cup B} & = & \{x | x \notin A \cup B\} & \text{[Def of Comp.]} \\
& = & \{x | \neg(x \in A \cup B)\} & \text{[Def of } \neg\text{]} \\
& = & \{x | \neg((x \in A) \vee (x \in B))\} & \text{[Def. of } \cup\text{]} \\
& = & \{x | \neg(x \in A) \wedge \neg(x \in B)\} & \text{[De Morgan]} \\
& = & \{x | (x \notin A) \wedge (x \notin B)\} & \text{[Def of } \neg\text{]} \\
& = & \{x | (x \in \overline{A}) \wedge (x \in \overline{B})\} & \text{[Def of Comp.]} \\
& = & \{x | x \in \overline{A} \cap \overline{B}\} & \text{[Def of } \cap\text{.]}
\end{array}
$$

Final Set Operator: Cartesian Product

Definition: *Ordered Pair*

An ordered pair is a group of two items (a, b) such that $(a, b) \neq (b, a)$ unless $a = b$.

Definition: *Ordered n-Tuple*

An ordered tuple is an ordered collection of n items $(a_i, a_2, \ldots, a_n)$ with a_i as its first element, a_2 as its second element, …, and a_n as its last (n^{th}) element.

Example:

$(1,2)$ is a different ordered pair than $(2,1)$

$\Rightarrow$ Remember: An ordered pair is ***not*** a set

(But you ***can*** create a set of ordered pairs!)

Final Set Operator: Cartesian Product

Definition: *Cartesian product*

The Cartesian Product of sets A and B ($A \times B$) is the set of all ordered pairs (a, b), $a \in A$, $b \in B$.

Or $X \times Y \equiv \{(x, y) \mid x \in X \wedge y \in Y\}$

Example:

$$A = \{\square, \triangle\}, \ B = \{r, s\}$$

$$A \times B = \{(\square, r), (\square, s), (\triangle, r), (\triangle, s)\}$$

$$B \times A = \{(r, \square), (s, \square), (r, \triangle), (s, \triangle)\}$$

Notes:

$A \times B \neq B \times A$, in general

$|A \times B| = |A| \cdot |B|$

Computer Representation of Sets

- Bit Vectors: One position per element in $\mathscr{U}$.

  # of bits = $|\mathscr{U}|$

Let $\mathscr{U} = \{a, b, c, d, e, f\}$

$$A = \{b, c, e\} \Rightarrow 011010$$

$$B = \{a, c, e, f\} \Rightarrow 101011$$

$$\overline{A} \Rightarrow \overline{011010} = 100101 \quad (\{a, d, f\})$$

$$A \cup B \quad \Rightarrow \quad \begin{array}{cr} & 011010 \\ \vee & 101011 \\ \hline & 111011 \end{array} \qquad A \cap B \quad \Rightarrow \quad \begin{array}{cr} & 011010 \\ \wedge & 101011 \\ \hline & 001010 \end{array}$$