CSc 245 Discrete Structures - Summer 2021

Quiz #6

Due: July 27th, 2021 by 11:59 pm (MST)

Solutions

1. (10 points) Consider the following sequence $\{s_n\}_{n=1}^{\infty}$ of integers: $1, 9, 25, 49, 81, 121, \ldots$. (Note, the sequence starts at $n = 1$ not $n = 0$).

   (a) (3 points) Give a simple function $f(n)$ such that $f(n) = s_n$ for $n \in \mathbb{Z}^+$.

   $f(n) = (2n - 1)^2$

   (b) (1 point) Using your answer to (a), give s_{10} and s_{13}.

   $s_{10} = 19^2 = 361$
   $s_{13} = 25^2 = 625$

   (c) (6 points) Prove, using weak induction, that $\sum_{i=1}^{n} f(i) = \frac{4n^3-n}{3}$ where $n \in \mathbb{Z}^+$ and $f(n)$ is the function you identified in (a).

   Proof (by induction): $f(n) = (2n - 1)^2$

   Base case: Let $n = 1$. $\sum_{i=1}^{n}(2i-1)^2 = (2(1)-1)^2 = 1$. $\frac{4n^3-n}{3} = \frac{4(1)-1}{3} = 1$. Thus our base case is true.

   Inductive Step: If $\sum_{i=1}^{k} f(i) = \frac{4k^3-k}{3}$, then $\sum_{i=1}^{k+1} f(i) = \frac{4(k+1)^3-(k+1)}{3}$.

   $$\begin{aligned}\sum_{i=1}^{k+1} f(i) &= \sum_{i=1}^{k} f(i) + f(k+1)\\ &= \frac{4k^3-k}{3} + f(k+1) \text{ (by our inductive hypothesis)}\\ &= \frac{4k^3-k}{3} + (2(k+1)-1)^2\\ &= \frac{4k^3-k}{3} + (2k+1)^2\\ &= \frac{4k^3-k}{3} + (4k^2+4k+1)\\ &= \frac{4k^3-k+3(4k^2+4k+1)}{3}\\ &= \frac{4k^3-k+12k^2+12k+3}{3}\end{aligned}$$

   We know that $(k+1)^3 = k^3 + 3k^2 + 3k + 1$
   We can modify our numerator as follows:
   $4k^3 - k + 12k^2 + 12k + 3 + 1 - 1$
   $= 4(k^3 + 3k^2 + 3k + 1) - k - 1$
   $= 4(k^3 + 3k^2 + 3k + 1) - (k + 1)$
   $= 4(k+1)^3 - (k+1)$
   So our fraction becomes:
   $\frac{4k^3-k+12k^2+12k+3}{3}$
   $= \frac{4(k+1)^3-(k+1)}{3}$
   Thus, $\sum_{i=1}^{k+1} f(i) = \frac{4(k+1)^3-(k+1)}{3}$. Therefore $\sum_{i=1}^{n} f(i) = \frac{4n^3-n}{3}$ where $n \in \mathbb{Z}^+$.