CSc 245 Discrete Structures - Summer 2021

Homework #4

Due: July 2nd, 2021 by 11:59 p.m (MST).

Solutions

Instructions:

1. **Homework assignments are to be completed individually, not in groups.**
2. If you need help, take advantage of Piazza and office hours.
3. Assignments are to be submitted in PDF form via Gradescope. They may be typed (which is preferable and strongly recommended) or handwritten with each page scanned or photographed and compiled into a single PDF.
4. If you choose to handwritten your assignments, please write neatly. Illegible assignments may not be graded.
5. Extra credit will be given for typed homework. To make this easier, a `Latex` template will be provided for each assignment.
6. Show your work (when appropriate) for partial credit!

1. Prove or disprove: If the average of 4 distinct integers is 10, at least one of the 4 integers must be greater than 12.

   The average of 8,9,11,12 is $(8 + 9 + 11 + 12)/4 = 10$

2. (6 points) Prove the following conjecture: for all real numbers a,b, and c, max(a,max(b,c))=max(max(a,b),c).

   Proposition: For all real numbers a,b, and c, max(a,max(b,c))=max(max(a,b),c).

   Proof (by cases):

   Case 1: $a \leq b \leq c$. Assume $a \leq b \leq c$.

   $\max(b,c) = c$, $\max(a,\max(b,c)) = \max(a,c) = c$.

   $\max(a,b) = b$, $\max(\max(a,b),c) = \max(b,c) = c$

   Thus, in this case, $\max(a,\max(b,c))=\max(\max(a,b),c)$

   Case 2: $a \leq c \leq b$. Assume $a \leq c \leq b$.

   $\max(b,c) = b$, $\max(a,\max(b,c)) = \max(a,b) = b$.

   $\max(a,b) = b$, $\max(\max(a,b),c) = \max(b,c) = b$

   Thus, in this case, $\max(a,\max(b,c))=\max(\max(a,b),c)$

   Case 3: $b \leq a \leq c$. Assume $b \leq a \leq c$.

   $\max(b,c) = c$, $\max(a,\max(b,c)) = \max(a,c) = c$.

   $\max(a,b) = a$, $\max(\max(a,b),c) = \max(a,c) = c$

   Thus, in this case, $\max(a,\max(b,c))=\max(\max(a,b),c)$

   Case 4: $b \leq c \leq a$. Assume $b \leq c \leq a$.

   $\max(b,c) = c$, $\max(a,\max(b,c)) = \max(a,c) = a$.

   $\max(a,b) = a$, $\max(\max(a,b),c) = \max(a,c) = a$

   Thus, in this case, $\max(a,\max(b,c))=\max(\max(a,b),c)$

Case 5: $c \leq a \leq b$. Assume $c \leq a \leq b$.

max(b,c) = b, max(a,max(b,c)) = max(a,b) = b.

max(a,b) = b, max(max(a,b),c) = max(b,c) = b

Thus, in this case, max(a,max(b,c))=max(max(a,b),c)

Case 6: $c \leq b \leq a$. Assume $c \leq b \leq a$.

max(b,c) = b, max(a,max(b,c)) = max(a,b) = a.

max(a,b) = a, max(max(a,b),c) = max(a,c) = a

Thus, in this case, max(a,max(b,c))=max(max(a,b),c)

Therefore, when a, b, and c, are real numbers, max(a,max(b,c))=max(max(a,b),c).

3. You have 6 colors of socks in your drawer. Prove, using contradiction, that if you pick 19 socks, you must have at least one matching pair of socks.

   Proposition: If we have 6 colors of socks and pick 19 socks, then we must have at least two pairs of one color.

   Proof (by contradiction) Assume not. Assume that we have 19 socks but do not have two pairs of one color. This means that we have at most 3 socks of each color. Let s_i be the number of socks we have of color i, where $1 \leq i \leq 6$.

   $\sum_{i=1}^{6} S_i \leq \sum_{i=1}^{6} 3 = 18$ This contradicts our assumption that we picked 19 socks.

   Therefore, if we have 6 colors of socks in our drawer and pick out 19 socks, then we must have at least two pairs of one color.

4. Prove the following conjecture: For any two integers m and n, $m - n$ is even if and only if $m^2 - n^2$ is even. Because this is a biconditional proposition (if and only if), be sure to prove both directions.

   Proposition: For any two integers m and n, $m - n$ is even if and only if $m^2 - n^2$ is even.

   Proof:

   ($\rightarrow$, direct): $m - n$ is only even when m and n are either both even or they are both odd. If the one is even and one is odd, then their difference will be odd.

   Case 1: Let m be even and n be even. By definition, $\exists k$ s.t. $m = 2k$ and $\exists l$ s.t. $n = 2l$

   $m - n = 2k - 2l = 2(k - l)$ which satisfies our antecedent.

   $m^2 - n^2 = (2k)^2 - (2l)^2 = 4k^2 - 4l^2 = 2(2k^2 - 2l^2)$. Thus $m^2 - n^2$, is even.

   Case 2: Let m be odd and n be odd. By definition, $\exists k$ s.t. $m = 2k + 1$ and $\exists l$ s.t. $n = 2l + 1$

   $m - n = 2k + 1 - (2l + 1) = 2k - 2l = 2(k - l)$ which satisfies our antecedent.

   $m^2 - n^2 = (2k + 1)^2 - (2l + 1)^2 = 4k^2 + 4k + 1 - (4l^2 + 4l + 1) = 2(2k^2 + 2k - 2l^2 - 2l)$. Thus $m^2 - n^2$, is even.

   ($\leftarrow$, contraposition): To prove by contraposition, we are going to prove that if $m - n$ is odd, then $m^2 - n^2$ is odd. We know from the previous part that $m - n$ is odd only when either m is odd and n is even or m is even and n is odd.

   Case 1: Let m be even and n be odd. By definition, $\exists k$ s.t. $m = 2k$ and $\exists l$ s.t. $n = 2l + 1$

   $m - n = 2k - (2l + 1) = 2(k - l) - 1$ which is odd and satisfies our antecedent.

   $m^2 - n^2 = (2k)^2 - (2l + 1)^2 = 4k^2 - (4l^2 + 4l + 1) = 2(2k^2 - 2l^2 + 4l) - 1 = 2(2k^2 - 2l^2 + 4l - 1) + 1$. Thus,$m^2 - n^2$ has the form $2j + 1$ and is odd.

   Case 2: Let m be odd and n be even. By definition, $\exists k$ s.t. $m = 2k + 1$ and $\exists l$ s.t. $n = 2l$

   $m - n = 2k + 1 - (2l) = 2k + 1 - 2l = 2(k - l) + 1$ which is odd and satisfies our antecedent.

$m^2 - n^2 = (2k+1)^2 - (2l)^2 = 4k^2 + 4k + 1 - 4l^2 = 2(2k^2 + 2k - 2l^2) + 1$. Thus,$m^2 - n^2$ has the form $2j + 1$ and is odd.

Therefore, if $m^2 - n^2$ is even, then $m - n$ is even.

Thus, for integers m and n, $m - n$ is even if and only if $m^2 - n^2$ is even.

5. Complete the following proof.

   Proposition: If $3|a^2$, then $3|a$.

   Proof: (By contradiction) Assume $3|a^2$ but $3 \not| a$.

   By definition of the divides operator, $a^2 \bmod 3 = 0$. We know that when we divide an integer by 3, there are only three possible remainders: $0, 1$, or 2 (otherwise, we could have divided out an additional 3). Thus, $a\%3$ will be either $0, 1$, or 2. However, since $3 \not| a$, $a\%3$ can only be 1 or 2.

   Assume $a\%3 = 1$. Thus, $a = 3n + 1$. $a^2 = 9n^2 + 6n + 1$. [Complete the proof starting here.]

   We can rewrite this as $a^2 = 3(3n^2 + 2n) + 1$. This is not divisible by 3 because it has the form $3k+1$ ($k = 3n^2 + 2n$) which we know by the definition of integer division that this will have a remainder of 1 when divided by 3. Thus this contradicts or assumption that $3|a^2$.

   Assume $a\%3 = 2$. Thus, $a = 3n + 2$ which means $a^2 = 9n^2 + 12n + 4$. We can rewrite this as $a = 3(3n^2 + 4n + 1) + 1$. Just as in the previous case, by definition of integer division a^2 is not divisible by 3. This contradicts our assumption that $3|a^2$.

   Thus, if $3|a^2$, then $3|a$.

6. Prove, by contradiction, that $\sqrt{3}$ is irrational. (Hint: use the result of the proof in the previous question).

   Proposition: $\sqrt{3}$ is irrational.

   Proof (by contradiction): Assume not. Assume $\sqrt{3}$ is rational.

   If $\sqrt{3}$ is rational, $\exists p, q \in \mathbb{Z}$ such that $\sqrt{3} = \frac{p}{q}$ where $\frac{p}{q}$ is in lowest terms.

   This means that $3 = \frac{p^2}{q^2}$.

   We can rewrite this as $3q^2 = p^2$ From the previous question, we know that since $3|p^2$, then $3|p$. So $p = 3k$ where $k \in \mathbb{Z}$. We can then rewrite p^2 as $9k^2$. $3q^2 = 9k^2$. This gives us $q^2 = 3k^2$. Again, using the proof in the last question, we know that since $3|q^2$, that $3|q$. However, this means that both p and q are divisible by 3 which means that they were not in lowest terms. This contradicts our assumption that they are in lowest terms.

   Thus $\sqrt{3}$ is irrational.

7. Prove, by contraposition, that if $x^3 - 1$ is even, then x is odd.

   Proposition: If $x^3 - 1$ is even, then x is odd.

   Proof (by contraposition): Using contraposition, we want to show that if x is even then $x^3 - 1$ is odd. If x is even, we know that $\exists k$ s.t. $x = 2k$. Thus, $x^3 - 1 = (2k)^3 - 1 = 8k^3 - 1$. To get this into our odd number form (of $2m + 1$), we add and subtract 2 from the equation to get $8k^3 - 1 + 2 - 2 = (8k^3 - 2) + 1 = 2(4k^3 - 1) + 1$. Since k is an integer, then $4k^3 - 1$ is an integer. Thus $x^3 - 1$ has the form $2m + 1$ where $m = 4k^3 - 1$ which means it is odd.

   Therefore, if $x^3 - 1$ is even, then x is odd.

8. Explain what is wrong with this proof.

   Proposition: If n and m are even integers, then $4|(n + m)$.

Proof: (Direct) Assume n and m are even.

By definition of an even number, $\exists k$ such that $n = 2k$. and $\exists k$ such that $m = 2k$.

Thus, $n + m = 2k + 2k = 4k$.

We know that $(n + m) \bmod 4 = 0$ because $n + m = 4k$ and $4k$ is clearly a multiple of 4

Because $(n + m) \bmod 4 = 0$, we know from the definition of the divides operator that $4|(n + m)$.

Therefore, if n and m are even integers, then $n + m$ is divisible by 4.

The same variable k is used for both n and m. However, the proposition just states that they are even integers, not that they are the same even integers.