CSc 245 Discrete Structures - Summer 2021

Homework #5

Due: July 16th, 2021 by 11:59 p.m (MST).

(64 points)

Solutions

Instructions:

1. **Homework assignments are to be completed individually, not in groups.**
2. If you need help, take advantage of Piazza and office hours.
3. Assignments are to be submitted in PDF form via Gradescope. They may be typed (which is preferable and strongly recommended) or handwritten with each page scanned or photographed and compiled into a single PDF.
4. If you choose to handwritten your assignments, please write neatly. Illegible assignments may not be graded.
5. Extra credit will be given for typed homework. To make this easier, a `Latex` template will be provided for each assignment.
6. Show your work (when appropriate) for partial credit!

1. (6 points) Determine if each of the following relations are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

   (a) $\{(a, b) \mid a$ is taking fewer courses than b, $a, b \in$Students$\}$

   irreflexive, antisymmetric, transitive

   (b) $\{(x, y) \mid x * y > 0,\ x \in \mathbb{Z}\}$

   reflexive, symmetric, transitive

   (c) $\{(x, y) \mid x = 2,\ x, y \in \mathbb{Z}\}$

   antisymmetric, transitive

2. (6 points) Let $R = \{(2, 1), (3, 1), (2, 2), (1, 3)\}$ on $\{1, 2, 3\}$ and $S = \{(1, 2), (3, 3), (2, 1), (1, 3), (4, 1)\}$ on $\{1, 2, 3, 4\}$. Perform the following operations:

   (a) $R \cap S$

   $R \cap S = \{(2, 1), (1, 3)\}$

   (b) $R - S$

   $R - S = \{(3, 1), (2, 2)\}$

(c) $R \circ S$

$R \circ S = \{(1,1),(1,2),(3,1),(2,3),(4,3)\}$

3. (6 points) Given the following relations, perform each specified operation.
$P = \{(x, y) \mid x = 2y \ x, y \in \mathbb{Z}\}$,
$Q = \{(x, y) \mid x = 2y + 1 \ x, y \in \mathbb{Z}\}$,
$R = \{(x, y) \mid xy > 0 \ x, y \in \mathbb{R}\}$, and
$S = \{(x, y) \mid xy < 0 \ x, y \in \mathbb{R}\}$

(a) $P \cap Q$

$P \cap Q = \emptyset$

(b) $P \cup Q$

$P \cap Q = \mathbb{Z} \times \mathbb{Z}$

(c) $R \circ S$

$R \circ S = \{(x, y) \mid xy < 0, \ x, y \in \mathbb{R}\}$

4. (4 points) For each of the following relations, give the matrix representation for that relation. Then, using the matrix, determine if the relation is reflexive, symmetric, antisymmetric, and/or transitive. Show your work.

(a) $R = \{(1,1),(1,2),(2,2),(3,1),(3,3)\}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

It is reflexive (all 1's all the diagonal).

It is not symmetric, because it is missing (2,1) and (1,3)

It is antisymmetric because it does not have (2,1) or (1,3).

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

It is not transitive because we have (2,1) and (1,3) but not (2,3).

(b) $R = \{(x, y) \mid x \% y = 0, \ x, y \in H\}$ on the set $H = \{1, 2, 3, 4\}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

It is reflexive (all 1's all the diagonal).

It is not symmetric, because it is missing (1,2), (1,3), (1,4) and (2,4)

It is antisymmetric because it does not have (1,2), (1,3), (1,4) or (2,4).

$$\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&0&1\end{bmatrix} \cdot \begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&0&1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&0&1\end{bmatrix}$$

It is transitive because there are ones at every position in the boolean product where there are ones in the original matrix.

5. (6 points) In lecture we learned how to determine properties of relations using matrices. We are also able to use matrices to find the union, intersection and composite of two relations. These three operations on relations correspond to the join, meet, and boolean product operators of zero-one matrices. For two relations A and B, we can find their union by using the "join" operator on their corresponding matrices ($M_A \wedge M_B$), their intersection by using the "meet" operator on their corresponding matrices ($M_A \vee M_B$), and their composite ($A \circ B$) by taking the boolean product of their corresponding matrices ($M_B \odot M_A$, note the order of the matrices). Let R and S be the relations on the set $\{a, b, c\}$ corresponding to the below matrices. Use the below matrices to perform the following operations on the relations R and S.

$$M_R = \begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix} \quad M_S = \begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix}$$

(a) $R \cup S$

This problem got messed up. I made a mistake with my symbols in the description so either of these two answers will suffice (the first one is the correct one).

$$\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix} \vee \begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix} \wedge \begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

(b) $R \cap S$

This problem got messed up. I made a mistake with my symbols in the description so either of these two answers will suffice (the second one is the correct one).

$$\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix} \vee \begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

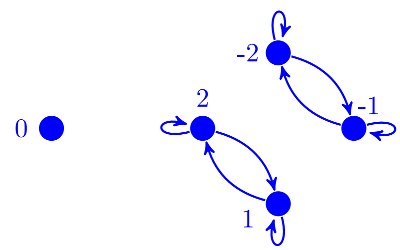
$$\begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix} \wedge \begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

(c) $R \circ S$

$$\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix} \odot \begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix} = \begin{bmatrix}1&1&0\\1&1&1\\1&1&1\end{bmatrix}$$

6. (5 points) Let $R = \{(x, y), \mid xy > 0 \; x, y \in H\}$ on $H = \{-2, -1, 0, 1, 2\}$.

   (a) Draw the graph of the following relation.



   (b) Using the graph, determine if the relation is reflexive, irreflexive, symmetric, antisymmetric, and/or transitive. Briefly explain your answer.

   It is not reflexive (0 does not have a self loop).
   It is not irreflexive (-2,-1,1, and 2 have self loops)
   It is symmetric because all edges between two nodes have a corresponding edge in the other direction.
   It is not anytisymmetric (because it is symmetric as described above).
   It is transitive.
   $(1, 2)\&(2, 1) \Rightarrow (1, 1)$ which is in the relation.
   $(2, 1)\&(1, 2) \Rightarrow (2, 2)$ which is in the relation.
   $(-1, -2)\&(-2, -1) \Rightarrow (-1, -1)$ which is in the relation.
   $(-2, -1)\&(-1, -2) \Rightarrow (-2, -2)$ which is in the relation.

7. (6 points) Describe how we can use the graphs of two relations R and S to create a graph corresponding to each of the following operations.

   (a) Intersection

   The graph of the intersection will contain all edges that are in the graphs of both R and S.

   (b) Composite

   We can create the graph of the composite by checking each node in the following way:
   For a node n, look for all incoming edges into n in the graph of S, and look at all outgoing edges of n in R. In the graph for $R \circ S$, add an edge from the node corresponding to each incoming edge on n in S to each of the nodes corresponding to the outgoing edges of n in R.

   (c) Inverse

   We can create the graph of the inverse by flipping the direction on all edges.

8. (9 points) For each of the following, determine if it is an equivalence relation. If it is an equivalence relation, show that it satisfies all required properties. If it is not, state all properties of an equivalence relation that it lacks and explain why it does not satisfy them.

   (a) $\{(a, b) \mid a$ and b have the same birthday, $a, b \in$People$\}$

   It is an equivalence relation. It is reflexive because every person has the same birthday as themself. It is symmetric because if person a has the same birthday as person b, then person b must have the same birthday as person a. It is transitive because if person a has the same birthday as person b and person b has the same birthday as person c, then person a clearly has the same birthday as person c.

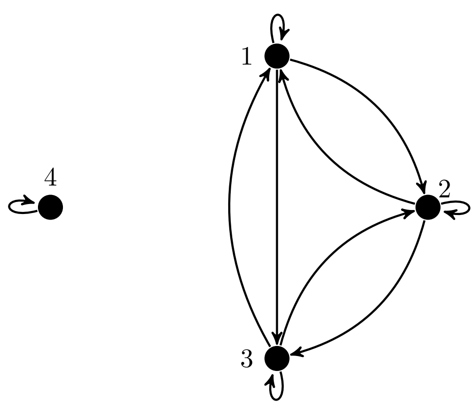
   (b) $\{(x, y) \mid xy > 0,\ x \in \mathbb{R}\}$

   It is not an equivalence relation. It is not reflexive because $0 \cdot 0$ is not greater than 0.

   (c) $\{(x, y) \mid x - y > 0\ x, y \in \mathbb{Z}\}$

   It is not an equivalence relation. It is not reflexive because $x - x$ is not greater than 0. It is not symmetric, because $y - x = -(x - y)$ so if $x - y = z$ where $z > 0$, then $y - x = -z$ must have $-z < 0$.

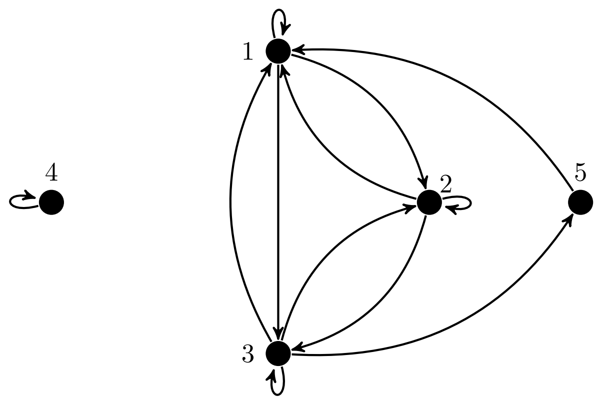
9. (4 points) Use the graph below to answer the following questions.



   (a) The above graph represents a relation on the set $\{1, 2, 3, 4\}$. What are the equivalence classes of the equivalence relation in the above graph?

   $[1] = \{1, 2, 3\}$, $[2] = \{1, 2, 3\}$, $[3] = \{1, 2, 3\}$, $[4] = \{4\}$

(b) We now modify our relation to expand its domain and codomain to be the set $\{1, 2, 3, 4, 5\}$ and adds some new pairs to the relation. Below is our updated graph. What edges must we add to this graph to make it an equivalence relation?



We must add the edges (5,5), (5,2), (2,5), (1,5), and (5,3).

10. (12 points) For each of the following, determine if it is (i) a weak partial ordering, (ii) a strict partial ordering and/or (iii) a total ordering. For each type of relation that it is, show that it satisfies the required properties. For any type that it is not, specify the properties it fails to satisfy and explain why it does not satisfy them.

(a) $\{(a, b) \mid a \text{ is older than } b \ a, b \in \text{People}\}$ on the set of all people in the world.

(i) It is not a weak partial order. It is not reflexive because person a cannot be older than themself.

(ii) It is a strong partial order. It is irreflexive because no person a can be older than themself. It is antisymmetric because if person a is older than b, then b cannot also be older than a. It is transitive because if b is older than a and c is older than b, then c must be older than a.

(iii) It is not a total order because it is not reflexive and is not comparable. It is not reflexive because person a cannot be older than themself. It is not comparable because if person a and person b are the same age, then the are not included in the relation and thus are not compared.

(b) $\{(x, y) \mid x \leq y,\ x, y \in \mathbb{Z}\}$ on the set $\mathbb{Z}$

(i) It is a weak partial order. It is reflexive because $x \leq x$ for any integer x. It is antisymmetric because if $x \leq y$ and $x \neq y$, then it must be the case that $x < y$. Since $x < y$, it cannot be the case that $x > y$. It is transitive because if $x \leq y$ and $y \leq z$, then x must be less than or equal to z.

(ii) It is a strong partial order. It is not irreflexive because $x \leq x$ for any integer x.

(iii) It is a total order. It is a weak partial order as shown in (i). It is comparable because for any pair of integers, we know how to order them using $\leq$.

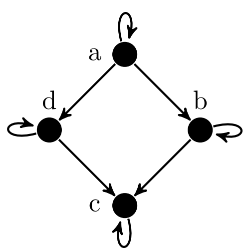
(c) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

(i) It is not a weak partial order. It is not reflexive becauseit has zeros all along the diagonal.

(ii) It is a strong partial order. It is irreflexive because it has zeros all along the diagonal. It is antisymmetric because for any position in the matrix, m_{ij}, if $m_{ij} = 1$, then $m_{ji} = 0$. It is transitive because when we take the boolean production of the matrix with itself, any position in the resulting matrix that has a 1, has a 1 in the original matrix.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(iii) It is not a total order because it is not reflexive. It is not reflexive because there are zeros all along the diagonal.

(d) 

(i) It is not a weak partial order. It is not transitive because we have (a, d) and (d, c) but we do not have (a, c) and we have (a, b) and (b, c) but we do not have (a, c).

(ii) It is a strong partial order. It is not irreflexive because every node has a self loop.

(iii) It is not a total order. It is not transitive because we have (a, d) and (d, c) but we do not have (a, c) and we have (a, b) and (b, c) but we do not have (a, c). It is not comparable because the pairs (a, c) and (c, a) are not in the relation and (d, b) and (b, d) are not in the relation.