CSc 245 Discrete Structures - Summer 2021

Homework #6

Due: July 23rd, 2021 by 11:59 p.m (MST).

(70 points)

Solutions

Instructions:

1. **Homework assignments are to be completed individually, not in groups.**
2. If you need help, take advantage of Piazza and office hours.
3. Assignments are to be submitted in PDF form via Gradescope. They may be typed (which is preferable and strongly recommended) or handwritten with each page scanned or photographed and compiled into a single PDF.
4. If you choose to handwritten your assignments, please write neatly. Illegible assignments may not be graded.
5. Extra credit will be given for typed homework. To make this easier, a `Latex` template will be provided for each assignment.
6. Show your work (when appropriate) for partial credit!

Part 1: Functions:

1. (4 points) For each of the following relations, determine if it is a function. Justify your answer.

   (a) $f(x) = \frac{1}{x^3}$ from $\mathbb{Z}$ to $\mathbb{R}$

   It is not a function because it is not defined for $x = 0$.

   (b) $f(x) = \sqrt{x}$ from $\mathbb{Z}^+$ to $\mathbb{R}$

   It is a function. Because x is positive, we can take the square root of it.

2. (4 points) For each of the following functions, give the domain and range.

   (a) $f(x) = x\%5$.

   Domain: $\mathbb{Z}$, Range: $\{0, 1, 2, 3, 4\}$

   (b) The function that, given a bit string, returns the length of the bit string.

   Domain: all possible bit strings, Range: $\mathbb{Z}^*$

3. (4 points) Evaluate each of the following:

   (a) $\lceil -3.5 \rceil$

   -3

   (b) $\lceil -2.2 * \lceil -\frac{5}{2} \rceil \rceil$

   $\lceil -2.2 * -2 \rceil = 5$

   (c) $\lfloor 2.99 \rfloor$

   2

   (d) $\lfloor \frac{4}{3} + \lceil \frac{1}{3} \rceil \rfloor$

   2

4. (6 points) For each of the following functions determine if it is injective. Justify your answer.

   (a) $f(n) = n^5$ from $\mathbb{Z}$ to $\mathbb{Z}$

   It is injective. Any integer in the range is mapped to by exactly one value of x: $x = \sqrt[5]{y}$. We know that because it's an odd power, when $x < 0$, it will return a negative value and when $x > 0$ it will return a positive.

   (b) $f(n) = |n| + 1$ from $\mathbb{Z}$ to $\mathbb{Z}^+$

   It is not injective. n and $-n$ will return the same value for $|n|$ and adding 2 to both of those values will keep them equal. $f(n) = f(-n)$.

   (c) $f(n) = n + 3$ from $\mathbb{Z}$ to $\mathbb{Z}$

   It is injective. Adding three to any integer will produce another unique integer. You cannot add 3 to 2 different integers to get the same result.

5. (6 points) Determine if each of the following functions from question 4 are surjective. Justify your answer.

   (a) $f(n) = n^5$ from $\mathbb{Z}$ to $\mathbb{Z}$

   It is not surjective as it does not map to every integer. For example, 2 is never mapped to by this function because $\sqrt[5]{2}$ is not an integer.

   (b) $f(n) = |n| + 1$ from $\mathbb{Z}$ to $\mathbb{Z}^+$

   It is surjective. Every positive integer, can be mapped to by $k+1$, where $k \geq 0$. And we know that $|n|$ where $n \in \mathbb{Z}$ will map to every non-negative integer, so we can see that $|n| + 1$ will map to all positive integers.

(c) $f(n) = n + 3$ from from $\mathbb{Z}$ to $\mathbb{Z}$

It is onto. All integers can be written as $x - 3$ where x is another integer. Thus, $f(x) = x + 3$ will map to all integers.

6. (6 points) Which of the functions from questions 4 and 5 are bijective? Briefly justify your answer.

   (a) $f(n) = n^5$ from $\mathbb{Z}$ to $\mathbb{Z}$

   It is not a bijection, because it is not onto.

   (b) $f(n) = |n| + 1$ from $\mathbb{Z}$ to $\mathbb{Z}^+$

   It is not a bijection because it is not 1-1

   (c) $f(n) = n + 3$ from from $\mathbb{Z}$ to $\mathbb{Z}$

   It is a bijection, because it is both 1-1 and onto.

7. (4 points) For each of the following, give an example of a function from $\mathbb{Z}^*$ to $\mathbb{Z}^*$ that satisfies the specified properties. You may not use the functions from the previous three problems.

   (a) One-to-one but not onto

   E.g. $f(x) = x^2$

   (b) Onto but not one-to-one

   E.g. $f(x) = \lfloor \frac{x}{2} \rfloor$

8. (4 points) Let C be the set of CS Faculty. Consider the following functions whose domains are C. Answer the following for each function: (i) Under what conditions is the function one-to-one? (ii) Under what conditions is the function onto?

   (a) The function f from C to O that assigns faculty to offices, where O is the set of offices on the 7th floor of Gould Simpson.

   (i) No faculty share an office.
   (ii) Every office is assigned to at least one faculty member.

   (b) The function g from C to S that assigns faculty to the courses they will teach this semester, where S is the set of courses offered by the CS department. Assume each faculty teaches exactly one course (otherwise it's not a function!).

   (i) Every course is taught by at most one faculty (i.e. no co-teaching).
   (ii) Every course offered by the CS department is taught this semester (at least one faculty member is assigned to it).

9. (4 points) Prove or disprove the following conjecture: $\lceil 3x \rceil = 3 \cdot \lceil x \rceil$.

Counterexample: Let $\lceil 3 \cdot 1.5 \rceil = \lceil 4.5 \rceil = 5$. $3 \cdot \lceil x \rceil = 3 \cdot 2 = 6$. Clearly,$\lceil 3x \rceil \neq 3 \cdot \lceil x \rceil$ when $x = 1.5$, so this conjecture is false.

10. (8 points) Prove the following conjecture: $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$. (Hint: Use the fact that, for any real number x, $x = m + d$ where $m \in \mathbb{Z}$, $d \in \mathbb{R}$ with $m \leq x < m + 1$ and $0 \leq d < 1$).

Proof (direct): Let $x = m + d$ where $m \in \mathbb{Z}$, $d \in \mathbb{R}$, $m \leq x < m + 1$ and $0 \leq d < 1$.

Let $y = n + c$ where $n \in \mathbb{Z}$, $c \in \mathbb{R}$, $n \leq y < n + 1$ and $0 \leq c < 1$.

From our definition of floor, we know that $\lfloor x \rfloor$ is the largest integer that does not exceed x. Thus, from our definition of the number x as $x = m + b$, we know that m is the largest integer that does not exceed x because $m \leq x < m + 1$.

As a result, $\lfloor x \rfloor = m$ and $\lfloor y \rfloor = n$. Thus, $\lfloor x \rfloor + \lfloor y \rfloor = m + n$.

Using our definitions, $x + y = (m + d) + (n + c) = (m + n) + (c + d)$. We know that $(m + n)$ will be an integer because m and n are both integers. We also know that $0 \leq (c + d) < 2$ because $0 \leq c < 1$ and $0 \leq d < 1$. We will consider 2 cases: when $0 \leq c + d < 1$ and when $1 \leq c + d < 2$.

Case 1: $0 \leq c + d < 1$. Let $x + y = p + b$, where $p = n + m$ and $b = c + d$.

We know, because $0 \leq c + d < 1$, that $0 \leq b < 1$. Subsequently we know that $p \leq p + b < p + 1$.

Because $x + y = p + b$, we get $p \leq x + y < p + 1$.

By definition of floor, $\lfloor x + y \rfloor = p$.

Thus, $\lfloor x + y \rfloor = n + m = \lfloor x \rfloor + \lfloor y \rfloor$ (as shown above).

So clearly, in this case, $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.

Case 2: $1 \leq c + d < 2$. We know that $x + y = n + m + c + d$.

We can rewrite this, to get $x + y = n + m + 1 + c + d - 1$ (which does not change the value of $x + y$).

Let $p = n + m + 1$, $b = c + d - 1$ and $x + y = p + b$.

We know that, because $1 \leq c + d < 2$, when we subtract one from $c + d$, we will get $0 \leq c + d - 1 < 1$.

Thus, $0 \leq b < 1$. Because $0 \leq b < 1$, we know that $p \leq p + b < p + 1$.

Subsequently, because $x + y = p + b$, we get $p \leq x + y < p + 1$.

By definition of the floor function, $\lfloor x + y \rfloor = p$.

Thus, $\lfloor x + y \rfloor = n + m + 1$

Clearly $n + m \leq n + m + 1$. Thus, in this case, $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.

Therefore, for any real number x, $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.

Part 2: Integers:

12. (4 points) For each of the following divisions, give the quotient and remainder.

(a) 85 divided by 12

Quotient: 7, Remainder: 1

(b) 230 divided by 15

Quotient: 15, Remainder: 5

13. (4 points) Give the prime factorization of the following integers.

    (a) 168

        $168 = 2^3 \cdot 3 \cdot 7$

    (b) 90

        $90 = 2 \cdot 3^2 \cdot 5$

14. (4 points) Determine whether each of theses sets are pairwise relatively prime. Show your work.

    (a) 60, 91, 11

        $60 = 2^2 \cdot 3 \cdot 5$
        $91 = 7 \cdot 13$
        GCD(11,91) = 1
        GCD(11,60) = 1
        GCD(60,91) = 1
        Therefore, they are pairwise relatively prime.

    (b) 105, 52, 13

        $105 = 3 \cdot 5 \cdot 7$
        $52 = 2^2 \cdot 13$
        GCD(105,52) = 1
        GCD(105,13) = 1
        GCD(52,13) = 13
        Therefore, they are not pairwise relatively prime.

15. (4 points) Give the GCD and LCM for each of the following pairs of integers.

    (a) $2^2 \cdot 3^2 \cdot 5 \cdot 13$, $2 \cdot 5^2 \cdot 7 \cdot 11$

        $\text{GCD}(2^2 \cdot 3^2 \cdot 5 \cdot 13,\ 2 \cdot 5^2 \cdot 7 \cdot 11) = 2 \cdot 5$
        $\text{LCM}(2^2 \cdot 3^2 \cdot 5 \cdot 13,\ 2 \cdot 5^2 \cdot 7 \cdot 11) = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$

    (b) $3^2 \cdot 7 \cdot 11$, $3 \cdot 5 \cdot 11^2$

        $\text{GCD}(3^2 \cdot 7 \cdot 11,\ 3 \cdot 5 \cdot 11^2) = 3 \cdot 11$
        $\text{LCM}(3^2 \cdot 7 \cdot 11,\ 3 \cdot 5 \cdot 11^2) = 3^2 \cdot 5 \cdot 7 \cdot 11^2$

16. (4 points) Given the following values of ab and either the GCD or the LCM, compute the LCM (if you were given the GCD) or the GCD (if you were given the LCM).

    (a) $ab = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11$, $\text{GCD}(a,b) = 2 * 3 * 7$

        $\text{LCM}(a,b)=2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$

    (b) $ab = 2^2 \cdot 13^2 \cdot 17$, $\text{LCM}(a,b) = 2 * 13^2 * 17$

        $\text{GCD}(a,b)=2$