CSc 245 Discrete Structures - Summer 2021

Homework #7

Due: July 30th, 2021 by 11:59 p.m (MST).
(75 points)

Instructions:

1. **Homework assignments are to be completed individually, not in groups.**
2. If you need help, take advantage of Piazza and office hours.
3. Assignments are to be submitted in PDF form via Gradescope. They may be typed (which is preferable and strongly recommended) or handwritten with each page scanned or photographed and compiled into a single PDF.
4. If you choose to handwritten your assignments, please write neatly. Illegible assignments may not be graded.
5. Extra credit will be given for typed homework. To make this easier, a `Latex` template will be provided for each assignment.
6. Show your work (when appropriate) for partial credit!
7. In your (inductive) proofs, be sure to do the following:
   - Start your proof with "Proof (style):" where style is the type of proof you are using.
   - Clearly label your base case and inductive step.
   - State the inductive hypothesis and the conjecture you are proving in the inductive step.
   - State any assumptions you are making.
   - Clearly define any variables used.
   - Conclude with "Therefore," and then restate the conjecture that you proved.

1. (2 points) Give the 9^{th} and 10^{th} terms of the following sequences.
   (a) $\{a_n\}_{n=1}^{\infty}$ where $a_n = (-1)^n + (-2)^{n-1}$
   (b) $\{b_n\}_{n=1}^{\infty}$ where $b_n = 12 + 3n + n^2$
2. (4 points) For each of the following sequences, provide a simple rule that describes them. Once you have found the rule, give the next two elements in the sequence. Note, the sequences go from $n = 1$ to infinity.
   (a) $1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, \ldots$
   (b) $0, 3, 8, 15, 24, 35, 48, 63 \ldots$
3. (3 points) For each of the following sequences, specify which of the following properties apply to them: increasing, non-decreasing, strictly increasing, decreasing, non-increasing, and/or strictly decreasing.
   (a) $9, 7, 5, 3, 1, -1, \ldots$
   (b) $1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, \ldots$
   (c) $1, 1, 1, 1, 1, 1, \ldots$

4. (6 points) For each of the following sequences, determine if they are arithmetic, geometric, both or neither. For those that are arithmetic and/or geometric, provide the common difference and/or ratio.

   (a) $9, 7, 5, 3, 1, -1, \ldots$

   (b) $2, 10, 50, 250, \ldots$

   (c) $729, 243, 81, 27, 9, \ldots$

5. (6 points) For each of the following sets, determine whether it is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a bijection between the set of positive integers and that set.

   (a) The set of integers that are perfect squares.

   (b) The set of real numbers that are the square roots of integers.

   (c) The set of real numbers between 0 and 1.

6. (9 points) Consider the conjecture: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for $n \geq 1$ where $n \in \mathbb{Z}^+$. This conjecture can be proven via induction.

   (a) State the base case for a proof by induction.

   (b) Show that the base case is true.

   (c) State the inductive hypothesis.

   (d) State the conjecture that we must prove in the inductive step.

   (e) Complete the proof of the inductive step.

7. (8 point) Let $s_n = s_{n-1} + 2n$ where $n \geq 1$ and $s_1 = 2$. Prove, using weak induction, that for any integer $n \geq 1$, $s_n = n(n+1)$

8. (8 points) Consider the conjecture $n! > 2^n$, where $n \in \mathbb{Z}^+$. Note, $n!$ is read "n factorial" and $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$

   (a) When proving this by induction, what should our base case be (i.e. for what value of n does this conjecture start being true)?

   (b) Prove that the conjecture holds for any value of n greater than or equal to the value found in part (a).

9. (8 points) Prove, using weak induction, that $3|(2^n + 1)$, for any *odd* integer $n \geq 1$.

10. (16 points) A baker is completing their monthly flour order. The flour mill only sells flour in 3kg bags and 7kg bags. The baker needs to order at least 12kgs of flour. Consider the following conjecture: the baker is able to order n kilograms of flour, in whole kilograms, for any $n \geq 12$, using a combination of 3kg and 7kg bags.

    (a) Prove the conjecture using weak induction.

    (b) Prove the conjecture using strong induction.

11. (5 points) Explain what is wrong with the following inductive proof.

Conjecture: For all positive integers n, $3^n = 3$.

Proof (by strong induction):

Base Case: Let $n = 1$. $3^1 = 3$. So our base case is true.

Inductive step: Assume that the conjecture holds for all $1 \le i \le k$ (i.e. for any $i \le k$, $3^i = 3$). We will show that if $\forall i$ where $1 \le i \le k$, $3^i = 3$ is true, then $3^{k+1} = 3$ must be true.

$$\begin{aligned}
3^{k+1} &= 3^k \cdot 3 && \text{(By algebra)} \\
&= 3^k \cdot \frac{3^k}{3^{k-1}} && \text{(By algebra)} \\
&= 3 \cdot \frac{3}{3} && \text{(By I.H on each term)} \\
&= 3 && \text{(simplifying)}
\end{aligned}$$

Therefore, because we have shown our base case and our inductive step to be true, for all positive integers n, $3^n = 3$.