CSc 245 Discrete Structures - Summer 2021

Homework #7

Due: July 30th, 2021 by 11:59 p.m (MST).

(75 points)

Solutions

Instructions:

1. **Homework assignments are to be completed individually, not in groups.**
2. If you need help, take advantage of Piazza and office hours.
3. Assignments are to be submitted in PDF form via Gradescope. They may be typed (which is preferable and strongly recommended) or handwritten with each page scanned or photographed and compiled into a single PDF.
4. If you choose to handwritten your assignments, please write neatly. Illegible assignments may not be graded.
5. Extra credit will be given for typed homework. To make this easier, a `Latex` template will be provided for each assignment.
6. Show your work (when appropriate) for partial credit!
7. In your (inductive) proofs, be sure to do the following:
   - Start your proof with "Proof (style):" where style is the type of proof you are using.
   - Clearly label your base case and inductive step.
   - State the inductive hypothesis and the conjecture you are proving in the inductive step.
   - State any assumptions you are making.
   - Clearly define any variables used.
   - Conclude with "Therefore," and then restate the conjecture that you proved.

1. (2 points) Give the 9^{th} and 10^{th} terms of the following sequences.

   (a) $\{a_n\}_{n=1}^{\infty}$ where $a_n = (-1)^n + (-2)^{n-1}$

   $a_9 = (-1)^9 + (-2)^8 = -1 + 256 = 255$
   $a_{10} = (-1)^{10} + (-2)^9 = 1 + (-512) = -511$

   (b) $\{b_n\}_{n=1}^{\infty}$ where $b_n = 12 + 3n + n^2$

   $b_9 = 12 + 3(9) + 9^2 = 12 + 27 + 81 = 120$
   $b_{10} = 12 + 3(10) + 10^2 = 12 + 30 + 100 = 142$

2. (4 points) For each of the following sequences, provide a simple rule that describes them. Once you have found the rule, give the next two elements in the sequence. Note, the sequences go from $n = 1$ to infinity.

   (a) $1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, \ldots$

   The sequence starting at one where every odd number repeats twice and evens repeat 3 times.
   $\{s_n\}_{n=1}^{\infty}$ where $s_n = \lceil \frac{2n}{5} \rceil$
   Next two elements: $7, 7$

   (b) $0, 3, 8, 15, 24, 35, 48, 63 \ldots$

   $\{s_n\}_{n=1}^{\infty}$ where $s_n = n^2 - 1$
   Next two elements: $80, 99$

3. (3 points) For each of the following sequences, specify which of the following properties apply to them: increasing, non-decreasing, strictly increasing, decreasing, non-increasing, and/or strictly decreasing.

   (a) $9, 7, 5, 3, 1, -1, \ldots$

   Decreasing, Strictly Decreasing, Non-increasing

   (b) $1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 6, \ldots$

   Non-decreasing, increasing

   (c) $1, 1, 1, 1, 1, 1, \ldots$

   Decreasing, Non-increasing, Non-decreasing, Increasing.

4. (6 points) For each of the following sequences, determine if they are arithmetic, geometric, both or neither. For those that are arithmetic and/or geometric, provide the common difference and/or ratio.

   (a) $9, 7, 5, 3, 1, -1, \ldots$

   Arithmetic, Common difference: -2

   (b) $2, 10, 50, 250, \ldots$

   Geometric, common ratio 5

   (c) $729, 243, 81, 27, 9, \ldots$

   Geometric, common ration $\frac{1}{9}$

5. (6 points) For each of the following sets, determine whether it is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a bijection between the set of positive integers and that set.

   (a) The set of integers that are perfect squares.

   It is countably infinite. A bijective function from $\mathbb{Z}^+$ to the set of integers that are perfect squares is $f(n) = (n-1)^2$ where $n \in \mathbb{Z}^+$. It is bijective because each positive integer maps to exactly one perfect square and each perfect square is mapped to by an integer.

   (b) The set of real numbers that are the square roots of integers.

   It is countably infinite. A bitjective function from $\mathbb{Z}^+$ to the set of real numbers that are square roots of integers is $f(n) = \sqrt{(n-1)}$ where $n \in \mathbb{Z}^+$. It is bijective because each positive integer has exactly one square root and each square root of an integer is clearly mapped to by an integer.

   (c) The set of real numbers between 0 and 1.

   It is uncountable.

6. (9 points) Consider the conjecture: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for $n \geq 1$ where $n \in \mathbb{Z}^+$. This conjecture can be proven via induction.

   (a) State the base case for a proof by induction

   The base case is $n = 1$ or $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1}$

   (b) Show that the base case is true.

   $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$ and $\frac{1}{1+1} = \frac{1}{2}$, so the base case is true.

   (c) State the inductive hypothesis.

   The inductive hypothesis states: $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ for $k \geq 1$.

   (d) State the conjecture that we must prove in the inductive step.

   If $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ for $k \geq 1$, then $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$

(e) Complete the proof of the inductive step.

Assume $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$.

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \text{ (via the I.H.)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{(k+1)}{(k+2)}
\end{aligned}$$

7. (8 point) Let $s_n = s_{n-1} + 2n$ where $n \geq 2$ and $s_1 = 2$. Prove, using weak induction, that for any integer $n \geq 1$, $s_n = n(n+1)$

Proof (weak induction):

Base case: $n = 1$. $s_1 = 2$. $1(1+1) = 2$ Thus $s_1 = 1(1+1)$ is true.

Inductive Step: If $s_k = k(k+1)$ for $k \geq 1$, then $s_{k+1} = (k+1)(k+2)$.

We know that $s_{k+1} = s_k + 2(k+1)$.

$$\begin{aligned}
s_{k+1} &= s_k + 2(k+1) \\
&= k(k+1) + 2(k+1) \\
&= (k+1)(k+2)
\end{aligned}$$

Thus, $s_{k+1} = (k+1)(k+2)$.

Therefore, for any integer $n \geq 1$, $s_n = n(n+1)$ where $s_n = s_{n-1} + 2n$.

8. (8 points) Consider the conjecture $n! > 2^n$, where $n \in \mathbb{Z}^+$. Note, $n!$ is read "n factorial" and $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$

   (a) When proving this by induction, what should our base case be (i.e. for what value of n does this conjecture start being true)?

   Our base case should be $n = 4$.

   (b) Prove that the conjecture holds for any value of n greater than or equal to the value found in part (a).

   Proof (weak induction):
   Base case: $n = 4$. $4! = 24$. $2^4 = 16$. $24 > 16$ so the base case is true.

   Inductive Step: If $k! > 2^k$ for $k \geq 4$, then $(k+1)! > 2^{k+1}$.
   From the inductive hypothesis, we know that $k! > 2^k$.
   We know that if we multiply an inequality by the same thing on both sides that it will not change.
   Thus, we can multiply both sides by $k+1$ to get $k!(k+1) > 2^k(k+1)$.
   $k!(k+1) = (k+1)!$ so we have $(k+1)! > 2^k(k+1)$.
   We know that $2^{k+1} = 2^k \cdot 2$. We also know that $k \geq 4$ which means that $k+1 > 4$ and so obviously, $k+1 > 2$. We can multiply both sides by 2^k to get $2^k(k+1) > 2^k \cdot 2$. Thus $2^k(k+1) > 2^{k+1}$.
   So by the transitive property, we get that $(k+1)! > 2^{k+1}$
   Therefore, when $n \geq 4$, $n! > 2^n$.

9. (8 points) Prove, using weak induction, that $3|(2^n + 1)$, for any *odd* integer $n \geq 1$.

   Proof (weak induction):
   Base case: $n = 1$. $2^(1) + 1 = 3$. Clearly, $3|3$ so our base case is true.

   Inductive Step: Let $n = 2k-1$ because this function generates the sequence of odd positive integers when $k \geq 1$.
   If $3|(2^{2k+1} - 1)$ for $k \geq 1$, then $3|(2^{2(k+1)-1} + 1)$. Note, we
   From our inductive hypothesis, we know that $3|(2^{2k-1}+1)$. We can rewrite this as $(2^{2k-1}+1) = 3m$ for some $m \in \mathbb{Z}$. Reworking this, we get $2^{2k-1} = 3m - 1$.

$$
\begin{aligned}
(2^{2(k+1)-1} + 1) &= 2^{2k+1} + 1 \\
&= 2^{(2k-1)} \cdot 2^2 + 1 \\
&= (3m-1)(4) + 1 \quad \text{(substituting from above)} \\
&= 12m - 4 + 1 \\
&= 12m - 3 \\
&= 3(4m-1)
\end{aligned}
$$

   Thus, we can see that $3|(2^{2(k+1)-1} + 1)$.
   Therefore, $3|(2^n + 1)$, for any *odd* integer $n \geq 1$.

10. (16 points) A baker is completing their monthly flour order. The flour mill only sells flour in 3kg bags and 7kg bags. The baker needs to order at least 12kgs of flour. Consider the following conjecture: the baker is able to order n kilograms of flour, in whole kilograms, for any $n \geq 12$, using a combination of 3kg and 7kg bags.

   (a) Prove the conjecture using weak induction.

   Proof (weak induction):

   Base case: $n = 12$. $12 = 3 \cdot 4$. Thus, we can clearly make 12kg using 4 3kg bags of flour.

   Inductive Step: If we can order k kilograms of flour where $k \geq 12$, then we can order $k + 1$ kilograms of flour.

   Assume we can order k kilograms of flour.

   If the order for k contains at least 2 3kg bags, then, to only add a single kg, we can simply remove 2 3kg bags and replace them with a 7kg bag. (i.e. If $k = 3m + 7l$ where $m \geq 2$, then $k + 1 = 3m + 7l + 1 = 3m - 6 + 7l + 7 = 3(m - 2) + 7(l + 1)$)

   If the order for k contains at least 2 7kg bags, then we can simply remove 2 7kg bags and replace them with 5 3kg bags. (i.e. If $k = 3m + 7l$ where $l \geq 2$, then $k + 1 = 3m + 7l + 1 = 3m + 15 + 7l - 14 = 3(m + 5) + 7(l - 2)$).

   In $n = 3m + 7l$, if $0 \leq l < 2$ and $0 \leq m < 2$, then we get $3m + 7l \leq 3(1) + 7(1) = 10 < 12$. Thus, it must be the case that at least one of $m > 2$ and $l > 2$ are true which means one of the above cases must apply for all $n \geq 12$.

   Therefore, the baker is able to order n kilograms of flour for any $n \geq 12$, using a combination of 3kg and 7kg bags.

   (b) Prove the conjecture using strong induction.

   Proof (strong induction):

   Base case: To prove this with strong induction, we must show $n = 12$, $n = 13$, and $n = 14$.

   $12 = 3(4) + 7(0)$.

   $13 = 3(2) + 7(1)$.

   $14 = 7(2)$.

   Thus, our base case is true.

   Inductive Step: If we can order i kilograms for any value of i where $12 \leq i < k$, then we can order k kilograms of flour.

   Assume $k > 14$. (We have already shown in our base case that it the conjecture is true for $12 \leq k \leq 14$).

   From our inductive hypothesis, we know that we can order $k-3$ kilograms. To get k kilograms, we can simply add a $3kg$ bag to our order.

   Thus, we can order k kilograms of flour.

   Therefore, the baker is able to order n kilograms of flour for any $n \geq 12$, using a combination of 3kg and 7kg bags.

11. (5 points) Explain what is wrong with the following inductive proof.

Conjecture: For all positive integers n, $3^n = 3$.

Proof (by strong induction):

Base Case: Let $n = 1$. $3^1 = 3$. So our base case is true.

Inductive step: Assume that the conjecture holds for all $1 \leq i \leq k$ (i.e. for any $i \leq k$, $3^i = 3$). We will show that if $\forall i$ where $1 \leq i \leq k$, $3^i = 3$ is true, then $3^{k+1} = 3$ must be true.

$$\begin{aligned} 3^{k+1} &= 3^k \cdot 3 && \text{(By algebra)} \\ &= 3^k \cdot \frac{3^k}{3^{k-1}} && \text{(By algebra)} \\ &= 3 \cdot \frac{3}{3} && \text{(By I.H on each term)} \\ &= 3 && \text{(simplifying)} \end{aligned}$$

Therefore, because we have shown our base case and our inductive step to be true, for all positive integers n, $3^n = 3$.

The problem is that our base case only shows the first case ($n = 1$). However, our inductive step relies on the prior 2 cases. So even though the inductive hypothesis assumes that it is true for all $1 \leq i \leq k$, using the $k - 1$ case is not valid because we have not shown enough cases in our basis step.