CSc 245 Discrete Structures - Summer 2021

Homework #8

Due: August 6th, 2021 by 11:59 p.m (MST).

(70 points)

Solutions

Instructions:

1. **Homework assignments are to be completed individually, not in groups.**
2. If you need help, take advantage of Piazza and office hours.
3. Assignments are to be submitted in PDF form via Gradescope. They may be typed (which is preferable and strongly recommended) or handwritten with each page scanned or photographed and compiled into a single PDF.
4. If you choose to handwritten your assignments, please write neatly. Illegible assignments may not be graded.
5. Extra credit will be given for typed homework. To make this easier, a `Latex` template will be provided for each assignment.
6. Show your work (when appropriate) for partial credit!

Multiplication and Addition Principles & Principle of Inclusion Exclusion

For the following problems, you should only use the multiplication principle, the addition principle and/or the principle inclusion exclusion to arrive at your answers. ***Give each answer as both an expression and an integer evaluation of the expression.***

1. Tina wants to add a PIN on her cellphone. She can add a PIN, consisting of digits 0-9, of any length of at least 4 to at most 17.

   (a) How many PINs of length 5 can Tina choose?

   $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^5 = 100,000$

   (b) Tina believes that a more secure password does not repeat any digits. How many PINs of length 5 can Tina create that do not repeat digits?

   $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30,240$

   (c) Tina's favorite numbers are 3 and 5. How many ways can she choose a PIN of length 5 that starts with either a 3 or a 5, and ends with the opposite digit (i.e. if it starts with 3, it ends with 5 and vice versa)?

   $2 \cdot 10 \cdot 10 \cdot 10 \cdot 1 = 2,000$

   (d) Tina does not like the number 7. How many ways can she choose a PIN of length 5 that does not contain a 7?

   $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^5 = 59,049$

(e) Tina can only remember numbers of at most 6 digits, thus she can only choose a PIN of length 4,5, or 6. How many ways can she choose a pin that she can remember?

$(10 \cdot 10 \cdot 10 \cdot 10) + (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) + (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) = 10^4 + 10^5 + 10^6 = 1,110,000$

2. A university generates 6 character student ID's in the following manner:

- The first character corresponds to the first letter of the students first name
- The second character corresponds to the first letter of the students last name
- The remaining 4 characters are decimal digits (0-9).

(a) How many unique ID's can the university generate?

$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26^2 \cdot 10^4 = 6,760,000$

(b) How many unique ID's can the university generate for a specific person?

$10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10,000$

(c) The university wants to ensure that they are able to create enough unique ID's. They modify their ID system slightly so that the two letters can now be placed as follows: (1) both letters are at the beginning, (2) the first letter is at the beginning and the second is at the end, or (3) both letters are at the end. (e.g. for a student with initials A.B., they could have ID's AB____, A____B, or ____AB). How many unique ID's can they create with this new system?

$(26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10) + (26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 26) + (10 \cdot 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26) = 3(26^2 \cdot 10^4) = 20,280,000$

3. Consider the sets D and C, where $|D| = 3$ and $|C| = 8$.

(a) How many unique ordered pairs (d, c) can we create where $d \in D$ and $c \in C$?

$3 \cdot 8 = 24$

(b) How many unique functions can we create from D to C?

$8 \cdot 8 \cdot 8 = 512$

(c) How many unique 1-1 functions can we create from D to C?

$8 \cdot 7 \cdot 6 = 336$

4. Linda is organizing her spice shelves in her kitchen. She has 12 spice bottles and 2 shelves to put them on. Linda wants to put 7 spices in a line on the first shelf.

(a) In how many different ways can she arrange 7 of her 12 spices on the first shelf?

$12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 3,991,680$

(b) Linda uses cinnamon and cardamom a lot, so she wants them to be on the first shelf. How many ways can she arrange 7 spices on her first shelf, if cardamom and cinnamon are both included?

There are 7 ways to add cardamom into the line. After adding cardamom, there a 6 places left where we can add cinnamon. After adding both of those, there are 5 spaces left for some of the 10 remaining spices. We can arrange those 10 into the remaining 5 spots in $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ ways. $7 \cdot 6 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 1,270,080$

(c) Linda decides instead that she wants either cinnamon or cardamom on the first shelf, but not both. How many ways can she arrange her first shelf so that it contains one of the two spices (cinnamon and cardamom), but no both.

There are 2 ways to choose one of the spices to add to the line and 7 ways to place it in the line. There are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ ways to add 5 of the remaining 10 spices to the line.

Thus there are $2 \cdot (7 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5) = 2,116,800$

5. Liza wants to buy some juice. There are 4 different juices that contain cranberry, 6 juices that contain apple, and 5 juices that contain grape. However, there are 2 juices that contain both apple and cranberry, 1 that contains both cranberry and grape, and 3 that contain both apple and grape. There is only 1 that contains all 3 - apple, grape and cranberry. How many different ways can Liza choose a juice?

Let C be the set of cranberry juices, A be the set of apple juices and G be the set of grape juices. Thus $|C| = 4$, $|A| = 6$, $|G| = 5$, $|C \cap A| = 2$, $|C \cap G| = 1$, $|A \cap G| = 3$, and $|C \cap A \cap G| = 1$.

The total number of ways to pick a juice is equal to $|C \cup A \cup G|$.

Using the principle of inclusion exclusion, we get

$|C \cup A \cup G| = |C| + |A| + |G| - |C \cap A| - |C \cap G| - |A \cap G| + |C \cap A \cap G| = 4+6+5-2-1-3+1 = 10$

6. Noel is choosing a moving to watch. He has 27 films to choose from. He knows that there are 20 films that are categorized as Action, Drama, and/or Comedy. Of those 20 films, he knows that 7 are action, 10 are comedy, and 11 are drama. He also knows that 2 are both action and comedy, 4 are action and drama, and 2 are action, drama and comedy.

(a) How many films does he can he choose that are both drama and comedy?

Let A be the set of action films, C be the set of comedy films and D be the set of dramas. $|A| = 7$, $|C| = 10$, and $|D| = 11$. $|A \cap C| = 2$, $|A \cap D| = 4$, $|A \cap C \cap D| = 2$, and $|A \cup C \cup D| = 20$.

$$\begin{aligned} |C \cap D| &= |A| + |B| + |C| - |A \cap C| - |A \cap D| + |A \cap C \cap D| - |A \cup C \cup D| \\ &= 7 + 10 + 11 - 2 - 4 + 2 - 20 = 4 \end{aligned}$$

(b) How many ways can Noel choose a movie that is **not** an action, drama, or comedy?

$\overline{|A \cup C \cup D|} = 27 - 20 - 7$

Pigeonhole

7. Prove that for any set of 16 (not necessarily consecutive) integers, there must be at least 4 with the same remainder when divided by 5.

   Proof (direct): Assume we have a set of 16 integers. We know that when dividing a number by 5, there are 5 possible remainders: $0, 1, 2, 3, 4$.

   Using the pigeonhole principle, we know that when we group our 16 integers by remainder, that one group must have at least $\lceil \frac{16}{5} \rceil$ members. $\lceil \frac{16}{5} \rceil = 4$. Thus, the one group of integers that share a remainder must have at least 4 elements in it.

   Therefore, for any set of 16 (not necessarily consecutive) integers, there must be at least 4 with the same remainder when divided by 5.

8. Assume that every student in the class has a favorite color from the following set: $C =$ {red, orange, yellow, green, blue, purple, pink}. At least how many students must be in the course to guarantee that 7 students have the same favorite color.

   We need to find the smallest integer n such that $\lceil \frac{n}{|C|} \rceil = 7$. The smallest of integer n where this is true is $n = (7-1) \cdot |C| + 1 = 6 \cdot 7 + 1 = 43$

Perm & Comb

9. Gene has 12 stuffed animals: 5 rabbits and 7 bears. Assume that all of the stuffed animals are distinguishable.

   (a) How many ways can they arrange their stuffed animals in a line?

   $12! = 479,001,600$

   (b) How many ways can they choose 3 rabbits and 4 bears from their stuffed animals?

   $\binom{5}{3} \cdot \binom{7}{4} = 350$

   (c) How many ways can they choose and arrange into a line 3 of the rabbits and 4 of the bears?

   $\binom{5}{3} \cdot \binom{7}{4} \cdot 7! = 1,764,000$

10. Joey has 8 coffee mugs and 4 pint glasses. Assume the mugs and glasses are distinguishable from each other.

    (a) How many ways can he arrange his mugs and glasses so that no two pint glass are placed next to each other?

    $8! \cdot \binom{9}{4} \cdot 4! = 121,927,680$

(b) How many ways can he arrange the mugs and glasses so that all pint glasses are placed together?

We can consider the pint glasses to be a single unit when arranging them with the mugs. So there are 9! ways to arrange the 8 mugs and the unit of pint glasses and then 4! ways to arrange the pint glasses.
$4! \cdot 9! = 8,709,120$

11. The CS department has 22 faculty, 8 lecturers, and 42 graduate students. How many ways can a committee be formed that has 5 faculty, 2 lecturers and 1 graduate student?

$\binom{22}{5} \cdot \binom{8}{2} \cdot 42 = 30,968,784$

12. We are creating decimal numbers of 8 digits. Note, the numbers may have leading zeros (e.g. 00000001 is a valid 8 digit number in this setting).

(a) How many ways can we create a number with exactly 3 even digits?

$\binom{8}{3} \cdot 5^3 \cdot 5^5 = 21,875,000$

(b) How many ways can we create a number with exactly 3 even digits without repeating digits?

$\binom{8}{3} \cdot P(5,3) \cdot P(5,5) = \binom{8}{3} \cdot 5 \cdot 4 \cdot 3 \cdot 5! = 403,200$

(c) How many ways can we create a number with fewer than 3 even digits?

$\sum_{i=0}^{2} \binom{8}{i} \cdot 5^i \cdot 5^{8-i} = 10,937,500 + 312,500 + 390,625 = 14,453,125$

13. How many subset of a size at most 4 exist from a set of 13 elements?

$\sum_{i=0}^{4} \binom{13}{i} = 715 + 286 + 78 + 13 + 1 = 1093$

Repetition/General Perm

14. The drug store sells 5 scents of bar soap, with plenty of each scent in stock. Charlotte needs to buy 14 bars of soap.

(a) How many ways can Charlotte buy 14 bars of soap?

Pips = 14 bars of soap, Pipes = 4 dividers between the 5 scents. Thus there are $\binom{18}{4}\binom{14}{14} = 3060$

(b) The store is having a sale where you get a discount on your whole bar soap purchase if you buy at least one of each scent of soap. Charlotte wants to take advantage of this discount. How many different ways can she choose 14 bars of soap so that she will get the discount?

We know there is only 1 way to pick the first 5 bars: simply pick one of each scent. So we need to pick the remaining 9 bars.
Pips = 9 bars of soap, Pipes = 4 dividers between the 5 scents. Thus there are $\binom{13}{4}\binom{9}{9} = 715$

15. Consider the word BOOKKEEPER.

(a) How many distinct strings can we make by rearranging the letters in BOOKKEEPER?

$\binom{10}{3}\binom{7}{2}\binom{5}{2} \cdot 3! = 151,200$

(b) How many distinct strings can we make by rearranging the letters in BOOKKEEPER if all K's must be adjacent to each other, all O's must be adjacent to each other, and all of the E's must be adjacent to each other?

$6! = 720$