CSc 245 Discrete Structures - Summer 2021

Quiz #3

Due: June 29th, 2021 by 11:59 pm (MST)

Solutions

1. (5 points) Show that when a and b are integers, $(a \mod 2)(b \mod 2) = ab \mod 2$

Proposition: $(a \mod 2)(b \mod 2) = ab \mod 2$

Proof (Direct): We know that we can rewrite any integer x in terms of another integer y using the quotient remainder form: $x = ym + r$ where m is the quotient, r is the remainder and $m, r \in \mathbb{Z}$. Using this definition, we can rewrite a and b in terms of the integer 2: $a = 2k + r_a$ and $b = 2l + r_b$ where $k, l \in \mathbb{Z}$ and $r_a, r_b \in \{0, 1\}$ (because those are the only possible remainders when dividing by 2).

Given these equations we get: $(a \text{ mod } 2) = r_a$ and $(b \text{ mod } 2) = r_b$ so $(a \text{ mod } 2)(b \text{ mod } 2) = r_a r_b$.

Now, multiplying $a * b$, we get:

$$\begin{aligned} ab &= (2k + r_a)(2l + r_b) \\ &= 4kl + r_a(2l) + r_b(2k) + r_a r_b \\ &= 2(2kl + lr_a + kr_b) + r_a r_b. \end{aligned}$$

Because $r_a, r_b \in \{0, 1\}$, we know that $r_a r_b$ will either be 0 or 1 which means that we cannot factor it by 2.

Since $r_a r_b$ cannot be factored by 2, we know from the definition of the modulus operator, that $(2(2kl + lr_a + kr_b) + r_a r_b) \text{ mod } 2 = r_a r_b$. Thus, when a and b are integers, $(a \text{ mod } 2)(b \text{ mod } 2) = ab \text{ mod } 2$ ■

2. (5 points) Prove that for all integers n, if n is odd then $(-1)^n = -1$.

Proposition: For all integers n, if n is odd then $(-1)^n = -1$

Proof (direct): Let n be an odd integer. By definition of odd integers, we know that $\exists k \in \mathbb{Z}$ s.t. $n = 2k + 1$. Substituting $2k + 1$ for n and using the laws of exponents we get

$$\begin{aligned} (-1)^n &= (-1)^{2k+1} \\ &= (-1)^{2k} * (-1)^1 \\ &= ((-1)^2)^k * (-1)^1 \\ &= (1)^k * (-1)^1 \\ &= (1) * (-1)^1 \\ &= (1) * (-1) \\ &= (-1) \end{aligned}$$

Therefore, when n is an odd integer, $(-1)^n = -1$ ■