CSc 245 - Introduction to Discrete Structures
Summer 2020

Final Solut ions

1. (5 Points) For each of the following, specify whether it is a proposition.

   (a) $5x > 0, x \in \mathbb{Z}$
   Not a proposition

   (b) Don't run with scissors.
   Not a proposition

   (c) Mushrooms are tasty.
   It is a proposition

   (d) 5+5 =2
   It is a proposition

   (e) This sentence is a lie.
   Not a proposition

2. (2 points) List the first four terms of the sequence $s_n = \lceil \frac{n}{2} \rceil + \frac{3n}{4}$ where $n \geq 5$.

$s_5 = \lceil \frac{5}{2} \rceil + \frac{3 \cdot 5}{4} = 3 + \frac{15}{4} = \frac{27}{4}$
$s_6 = \lceil \frac{6}{2} \rceil + \frac{3 \cdot 6}{4} = 3 + \frac{18}{4} = \frac{30}{4}$
$s_7 = \lceil \frac{7}{2} \rceil + \frac{3 \cdot 7}{4} = 4 + \frac{21}{4} = \frac{37}{4}$
$s_8 = \lceil \frac{8}{2} \rceil + \frac{3 \cdot 8}{4} = 4 + \frac{24}{4} = 40 = 10$

3. (4 points) Find the GCD and LCM of 54 and 72

$54 = 2 \cdot 3^3$

$72 = 2^3 \cdot 3^2$

$GCD(54, 72) = 2 \cdot 3^2 = 18$

$LCM(54, 72) = 2^3 \cdot 3^3 = 216$

4. (5 points) Write the truth table for $\neg(\neg p \vee q) \wedge \neg(p \rightarrow q)$. Be sure to include all columns.

p	q	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg(\neg p \vee q) \wedge \neg(p \rightarrow q)$
T	T	F	T	F	T	F	F
T	F	F	F	T	F	T	T
F	T	T	T	F	T	F	F
F	F	T	T	F	T	F	F

5. (4 points) Let $P(x, y) = x|y$. Express the truth value for the following two quantification's. Briefly explain your answer.

   (a) $\forall y \exists x P(x, y)$, $x, y \in \mathbb{Z}$

   True, every integer y is divisible by some integer x. (All integers are divisible by 1).

   (b) $\exists y \forall x P(x, y)$, $x, y \in \mathbb{Z}$

   False. The only integer that is divisible by every integer except itself is zero. However, since zero is in the domain of x, the statement must be false because 0 doesn't divide 0.

6. (6 points)Use rules of inference to show that conclusion follows from the given premises. First identify and label all predicates. Then convert the premises and conclusion to their corresponding logic statements using those predicates. Then apply rules of inference to the premises to reach the given conclusion. Note: The given premises alone <u>do not</u> give the conclusion, you must apply rules of inference to them to reach the conclusion.

   All dogs like treats.

   Rufus (a dog) likes to chew bones.

   $\therefore$ There is a dog who likes treats and chewing bones.

   $T(x)$: x likes treats, $x \in$ dogs

   $B(x)$: x likes to chew bones, $x \in$ dogs

   $\forall x\ T(x)$, $x \in$dogs

   $B(\text{Rufus})$

   $\therefore \exists x\ T(x) \wedge B(x)$, $x \in$ dogs

	(1)	$\forall x\ T(x)$, $x \in$dogs	(Given)
	(2)	$B(\text{Rufus})$	(Given)
	(3)	$T(\text{Rufus})$	(Universal instantiation of (1))
	(4)	$B(\text{Rufus}) \wedge T(\text{Rufus})$	(Conjunction of (2) and (3))
$\therefore$	(5)	$\exists x\ T(x) \wedge B(x)$, $x \in$ dogs	(Existential Generalization of (4))

Alternate Answer:

$D(x)$: x is a dog, $x \in$ animals

$T(x)$: x likes treats, $x \in$ animals

$B(x)$: x likes to chew bones, $x \in$ animals

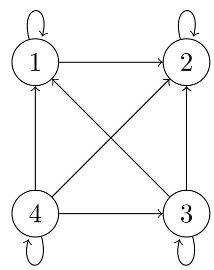
$\forall x\ D(x) \rightarrow T(x),\ \ x \in$animals

$D(\text{Rufus}) \wedge B(\text{Rufus})$

$\therefore \exists x\ D(x) \wedge T(x) \wedge B(x),\ \ x \in$ animals

	(1)	$\forall x\ D(x) \rightarrow T(x),\ \ x \in$animals	(Given)
	(2)	$D(\text{Rufus}) \wedge B(\text{Rufus})$	(Given)
	(3)	$D(\text{Rufus}) \rightarrow T(\text{Rufus})$	(Universal instantiation of (1))
	(4)	$D(\text{Rufus})$	(Simplification of (2))
	(5)	$T(\text{Rufus})$	(Modus Ponens of (3) and (4))
	(6)	$D(\text{Rufus}) \wedge B(\text{Rufus}) \wedge T(\text{Rufus})$	(Conjunction of (2) and (5))
$\therefore$	(7)	$\exists x\ D(x) \wedge T(x) \wedge B(x),\ \ x \in$ animals	(Existential Generalization of (6))

7. (12 points) Given the graph below of the relation R, answer the following:



(a) Give the matrix representation of the relation.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) Determine if the relation is a weak partial ordering, strong partial ordering and/or total ordering. Justify your answer.

Reflexive: yes

Antisymmetric: yes

Transitive: $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ Everywhere there is a 1 in the boolean product, has a 1 in the original matrix thus it is transitive.

Comparable: All pairs of elements are compared in the relation.

Therefore, it is a partial ordering and a total ordering.

(c) How can we change the graph to represent the inverse of the relation?

If we reverse the arrows, we get the inverse fo the relation.

(d) Is the relation a function? Explain why or why not.

It is not a function because some elements in the domain (1,3, and 4) map to multiple elements of the codomain which violates the definition of a function.

8. (8 points) For each of the following functions from $\mathbb{Z} \to \mathbb{Z}$, determine if they are injective, surjective, and/or bijective. Justify your answers.

(a) $f(x) = x^2 + 1$

It is not injective because $f(-1) = f(1) = 2$ so 2 is mapped to multiple times.

It is not surjective because the negative integers and zero are not mapped to by the function.

Because it isn't surjective and injective, it is also not bijective.

(b) $f(x) = x^3 + 8$

It is injective because no two values of x can map to the same value of y. $f(x_1) = f(x_2)$ means $x_1^3 + 8 = x_2^3 + 8$. Subtracting 8, we get $x_1^3 = x_2^3$. Since the cube of a positive is positive and the cube of a negative is negative, $x_1 = x_2$.

It is not surjective. $2 = x^3 + 8$ does not have an integer solution.

Because it is not surjective, it is not bijective.

(c) $f(x) = x - 2$

It is injective because no two values of x can map to the same value of y. $f(x_1) = f(x_2)$ means $x_1 - 8 = x_2 - 2$. Subtracting 2, we get $x_1 = x_2$. Thus, no two values of x can map to the same value of y.

It is surjective. Every integer is mapped to by the integer that is two less than it.

Because it is both injective and surjective, it is bijective.

(d) $f(x) = \lfloor \frac{x}{3} \rfloor$

It is not injective. $f(1) = f(2) = 1$.

It is surjective. Every integer y will be mapped to by at least one integer, $3y$.

It is not bijective because it is not injective.

9. (10 points) Use a proof by contradiction to show that if $n\%5 \neq 0$ then n is not the sum of five consecutive integers.

Proof (by contradiction)

Assume $n\%5 \neq 0$ but n is the sum of 5 consecutive integers.

n can be rewritten as $n = k + (k+1) + (k+2) + (k+3) + (k+4)$

Simplifying, we get $n = 5k + 10 = 5(k+2)$

This implies that n is divisible by 5 which contradicts the assumption that $n\%5 \neq 0$.

Therefore, if $n\%5 \neq 0$ then n is not the sum of five consecutive integers

10. (6 points) Determine if each of the following sets are countable or uncountable. Justify your answer. For those that are countable, create a bijective mapping from the set to either the positive integers or non-negative integers.

(a) The negative integers

Countable. We can map from the positive integers using the function $f(x) = -x$. This function is bijective because it is invertable: $-y = x$

(b) The even integers

Countable. We can map from the even integers to the nonnegative integers using the function

$$f(x) = \begin{cases} x & x >= 0 \\ -x - 1 & x < 0 \end{cases}$$

This gives pairings following the pattern: $\{(0,0), (-2,1), (2,2), (-4,3), (4,4), \dots\}$
This function is bijective because it is invertable:

$$f(y) = \begin{cases} y & y \text{ is even} \\ -(y+1) & y \text{ is odd} \end{cases}$$

(c) The real numbers between 0 and $\frac{2}{3}$

Uncountable. No function can map the real numbers between 0 and 2/3 to the positives.

11. (9 points) The animal shelter has 30 dogs. 6 of these dogs are part or whole Golden Retriever, 11 are part or whole Husky, and 8 are part or whole Great Dane. 2 of the dogs are both Golden Retriever and Husky and 3 of the dogs are both Husky and Great Dane. Only 1 of the dogs was a mix of Great Dane and Golden Retriever. None of the dogs are a mix of all three breeds.

    (a) How many dogs are whole Husky?

    $11 - 2 - 3 = 6$

    (b) What is the cardinality of the union of the sets of dogs that are part or whole Great Dane and part or whole Husky?

    $8 + 11 - 3 = 16$

    (c) How many dogs are neither Golden Retriever, Husky nor Great Dane?

    $30 - (8 + 11 + 6 - 2 - 3 - 1) = (30 - 19) = 11$

12. (10 points) A grocery store sells flour in 3lb and 8lb bags. Use induction to prove that a customer could buy any amount (in pounds) of flour greater than 13 lbs.

    Proof (strong induction):

    Let $P(n)$ be the statement n pounds of flour can be formed using 3lb and 8lb bags.

    Base Case:

    $P(14) = 3(2) + 8$ so it is True.

    $P(15) = 3(5)$ so it is True.

    $P(16) = 8(2)$ so it is True.

    Thus the base case is true.

    Inductive Case:. If $P(j)$ is true for all $16 \leq j < k$, then $P(k)$ is true.

    We can form k pounds by adding a 3 pound bag to $P(k-3)$. We know from our inductive hypothesis that $P(k-3)$ will be true because k is greater than 16 which means $k - 3 > 13$. Thus, $P(k) = P(k-3) +$ a 3lb bag which makes it true.

    Therefore, a customer could buy any amount (in pounds) of flour greater than 13 lbs using only 3 lb and 8 lb bags.

13. (18 points)Rodger's Board & Brew is a brewery that also has a collection of board games for patrons to play while visiting. They have 5 different categories of board games: Party, Strategy, Dice, Card, and Bluffing, and 6 types of beer: IPA, Amber, Sour, Wheat, Pale Ale, and Cream Ale. Patrons can either buy a beer or pay $5 to play up to 3 games. After 3 games, they must buy another beer or pay another $5 to play up to three more, and so on.

(a) David buys 2 beers to play 6 games. In how many ways can David select the 2 beers and 6 games?

There are 6 types of beer. Using the pips and pipes method, that gives us 5 dividers (pipes) and 2 beers (pips). Thus there is $\binom{7}{2}$ ways to choose the beers.
There are 5 types of games. Using the pips and pipes method, that gives us 4 dividers (pipes) and 6 games (pips). Thus there is $\binom{10}{6}$ ways to choose the games.
Thus, there are $\binom{7}{2}\binom{10}{6} = 21 \cdot 210 = 4410$ ways to choose 2 beers and 6 games.

(b) David likes strategy games but can play no more than 1 in a night. How many ways can he select 7 games given that he selects no more than 1 strategy game?

He plays 1 strategy game: There are 4 other game types. Using the pips and pipes method, that gives us 3 dividers (pipes) and 6 other games (pips). Thus there is $\binom{10}{6}$ ways to choose the rest of the games.
He does not play a strategy game: There are 4 other game types. Using the pips and pipes method, that gives us 3 dividers (pipes) and 7 games (pips). Thus there is $\binom{10}{7}$ ways to choose the games.
Thus, there are $\binom{10}{6} + \binom{10}{7} = 210 \cdot 120 = 330$ ways to choose 7 games if he chooses at most 1 strategy game.

(c) David is not a fan of bluffing games or sour beers. If he buys 3 beers to play 8 games, how many ways can he choose those 3 beers and 8 games if he excludes bluffing games and sour beers?

There are 5 types of beer excluding sours. Using the pips and pipes method, that gives us 4 dividers (pipes) and 3 beers (pips). Thus there is $\binom{7}{3}$ ways to choose the beers.
There are 4 types of games excluding bluffing games. Using the pips and pipes method, that gives us 3 dividers (pipes) and 8 games (pips). Thus there is $\binom{11}{8}$ ways to choose the games.
Thus, there are $\binom{7}{3}\binom{11}{8} = 35 \cdot 165 = 5775$ ways to choose32 beers and 8 games when we exclude sour beers and bluffing games.

(d) Lisa buys a 2 ounce pour of each beer so she can try all of them. How many ways can she arrange her drinks?

There are $6! = 720$ ways for Lisa to arrange her drinks.

(e) After tasting all of the beer she decides to get cans of 3 of them to take home with her. How many ways are there for her to select and arrange her three cans?

There are 6 types of beer. Using the pips and pipes method, that gives us 5 dividers (pipes) and 3 beers (pips). Thus there is $\binom{8}{3}$ ways to choose the cans of beer.

Once the cans are chosen, there are 3! ways to arrange them.
Thus there are $\binom{8}{3}3! = 56 \cdot 6 = 336$ ways to choose and arrange 3 cans.

(f) A large party arrives and is trying to decide what kind of game to play. Each category is voted for by at least 2 people. At least how many people are in the group?

Since there are 5 categories of games, and each category was voted for by at least 2 people, there are at least 10 people in the group.

14. (18 points) Algorithms

(a) Given the recursive algorithm below give the recursive definition.

```
subprogram triplesum (given: n)
        returns: sum from 1 through n of 3i

    if n is 1, return 3
    otherwise
        answer <-- triplesum(n-1)+3*n
        return answer
    end if

end subprogram
```

Basis Clause: triplesum(1) = 3
Inductive Clause: triplesum(n) = triplesum(n-1)+3n
Extremal Clause: n is an integer.

(b) Prove that the algorithm above returns $\sum_{i=1}^{n} 3i$.

Proof (weak induction):

Base Case: triplesum(1) = 3 = $\sum_{i=1}^{1} 3i$

Inductive Case: If triplesum(k) = $\sum_{i=1}^{k} 3i$, then triplesum(k+1) = $\sum_{i=1}^{k+1} 3i$.

We know from our recursive definition, that triplesum(k+1) = triplesum(k)+3(k+1).

Using the inductive hypothesis, we can substitute for triplesum(k) to get:

triplesum(k)+3(k+1) $= \sum_{i=1}^{k} 3i + 3(k+1)$.

The summation in $\sum_{i=1}^{k} 3i + 3(k+1)$ can absorb $3(k+1)$ to give $\sum_{i=1}^{k+1} 3i$.

This is our desired conclusion.

Thus, triplesum(n) defined in the algorithm above always returns $\sum_{i=1}^{n} 3i$.

15. (8 points)Recurrence Relations: Find the solution to the linear homogeneous recurrence relation with constant coefficient of degree 2: $R(n) = 4R(n-2)$ where $R(0) = 2$ and $R(1) = 4$

Using our steps from lecture, we first need to find the characteristic function: $w^2 - c_1 w - c_2$.

In our case, $c_1 = 0$ and $c_2 = 4$.

Our characteristic equation then becomes $w^2 - 4$.

The roots of this equation are $r_1 = 2$ and $r_2 = -2$.

Plugging these into our closed form expression, we get $R(n) = \alpha_1 2^n + \alpha_2(-2)^n$.

Next, we use our initial conditions to solve for α_1 and α_2.

$R(0) = \alpha_1 + \alpha_2 = 2$.

$R(2) = 2\alpha_1 - 2\alpha_2 = 4$.

Solving the first equation for α_1 we get $\alpha_1 = 2 - \alpha_2$.

Plugging that into our second equation, we get $2(2 - \alpha_2) - 2\alpha_2 = 4$.

Simplifying we get, $4 - 4\alpha_2 = 4$ which leads to $\alpha_2 = 0$

Plugging that back in to our equation for α_1 we get $\alpha_1 = 2 - 0 = 2$.

Thus, our closed form formula for $R(n)$ is $R(n) = 2(2^n) = 2^{n+1}$.