Finite Probability

Probability

Definition: *Probability*

The probability that a specific event will occur is the ratio of the number of occurrences of interest to the number of possible occurrences

- The occurrences of interest are called **events (E)**.
- The set of possible occurrences is the **Sample space (S)**.
- These are finites sets, hence the term *finite* probability
- The occurrence probability of an interest event:

$$p(E) = \frac{|E|}{|S|}$$

Probability

Please note: (a) $\forall e \in S,\ p(e) > 0$ (b) $\sum_{e \in S} p(e) = 1$

Example:

The probability of drawing a 10 from a 52-card deck:

$$p(E) = \frac{|E|}{|S|} = \frac{|\{10C, 10D, 10H, 10S\}|}{52} = \frac{1}{13}$$

Roll a pair of 6-sided, fair dice. What is the probability of one showing a 4 and the other either a 1 or a 6?

$$E = \{(4,1), (4,6), (1,4), (6,4)\}; \quad |E| = 4$$

$S =$ all possible rolls of two dice; $\quad |S| = 36$

Thus: $p(E) = \frac{|E|}{|S|} = \frac{4}{36} = \frac{1}{9}$

Applications of Counting to Probability

1. Probability of Winning the Powerball Lottery

To play: Choose 5 of 59 ‘white’ and 1 of 35 ‘red’ (Powerball) numbers. # of ways to play?

There are $\binom{59}{5}$ ways to choose the whites, and

$\binom{35}{1}$ ways to choose a red.

By the Multiplication Principle, there are

$\binom{59}{5} \cdot \binom{35}{1} = 175{,}223{,}510$ ways to play

$$p(\text{win}) = \frac{|E|}{|S|} = \frac{1}{175{,}223{,}510} \approx 5 \times 10^{-9} = 0.000000005$$

Applications of Counting to Probability

2. Principle of Inclusion-Exclusion

Recall: $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example:

54 students:	28 freshman	(13 men, 15 women)
	20 sophomores	(12 men, 8 women)
	6 juniors	(5 men, 1 women)

The probability of drawing a woman's name:

$$p(\text{woman}) = \frac{15+8+1}{54} = \frac{4}{9} = 0.\overline{4}$$

… of drawing either a woman (W) or a freshman (F)

$$p(W \cup F) = p(W) + p(F) - p(W \cap F)$$

$$= \frac{24}{54} + \frac{28}{54} - \frac{15}{54} = \frac{37}{54} = (1 - \frac{17}{54}\text{(fraction of male non-freshman))}$$

Probabilistic Reasoning

- Each drawer of a 3x2 dresser holds either a red or a blue T-shirt. One row of drawers has two red shirts, one row has two blue, and one row has one of each. You open one drawer and see a red T-shirt. What is the probability that the shirt in the other drawer is the same row is also red?

Probabilistic Reasoning

- One solution approach: Enumerate the possibilities. WLOG:

Dresser

R1	R2
B	B
R3	B

Open Drawer Containing	**Shirt Color in Other Drawer?**
R1	Red
R2	Red
R3	Blue

$$p(\textbf{The opposite drawer has red}) = \frac{2}{3}$$

Bertrand's Box Paradox (1889)

Probabilistic Reasoning

- Suppose you are on a game show, and you're given the choice of 3 doors: Behind one door is a car; behind the others goats. You pick a door, say No. 1, and the host who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you "Do you. Want to pick door No. 2?" Is it to your advantage to switch your choice?

From "Ask Marilyn", Parade, Sept. 9, 1990.

- Reference:
  - www.marilynvossavant.com/articles/gameshow.html

Care to Play?

URL: **`http://dcity.org/brain-games/three-door-dilemma`**

Probabilistic Reasoning

- But … why? Three views:

1. Enumerate the Possibilities

   WLOG, assume that we always pick door #1

   There are 3 possibilities:

Case	Door #1	Door #2	Door #3	Host Reveals	Switch?
(a)	Car	Goat	Goat	#2 or #3	You lose!
(b)	Goat	Car	Goat	#3	You wIn!
(c)	Goat	Goat	Car	#2	You wIn!

$$\therefore p(\textbf{win}) = \frac{2}{3}$$ **if you switch.**

Probabilistic Reasoning

2. Car / Not Car

   Consider a partitioning of the doors into two gropus: Selected and Unselected

   $p(\text{Car behind Selected}) = \frac{1}{3}$ and

   $p(\text{Car behind unselected}) = \frac{2}{3}$

   After the host reveals a goat, those probabilities have not changed!

   $p(\text{win without switching doors}) = \frac{1}{3}$ (stay with Selected)

   $p(\text{win after switching doors}) = \frac{2}{3}$ (move to Unselected group)

   $\Rightarrow$ The remaining closed door represents the Unselected group

Extra Slides (not covered)

Conditional Probability

Example:

If we have one fair 6-sided die:

$$p(6) = \frac{1}{6} \qquad p((6,6)) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Let E_1 be rolling a 6 on the 1st roll, & E_2 a 6 on the 2nd

There are 6 ways for each to occur, but what if E_1 has *already* occurred?

In that case: $p(E_2)$ still is $\frac{1}{6}$ because the first rolls does not influence the second.

But … what if the first <u>does</u> influence the second?

Conditional Probability

Definition: *Conditional Probability*

Let X and Y be events. The conditional probability of X given Y, denoted $p(X \mid Y)$, is $\dfrac{P(X \cap Y)}{p(Y)}$

Example:

Continuing the students example, what is $P(W \mid F)$?

$$p(W \mid F) = p(W \cap F)/P(F)$$

$$= \frac{15}{54} / \frac{28}{54} = \frac{15}{28} = 0.536\ldots$$

(Easy to see from 2 slides ago: 15 women of 18 freshman)

(Recall that $p(W) = \dfrac{24}{54} = \dfrac{4}{9} = 0.\overline{4}$)

Independence of Events

Recall: $p(A \mid B) = \dfrac{p(A \cap B)}{p(B)}$

Definition: *Independent*

Events A and B are independent if $p(A \mid B) = p(A)$

Example:

You flip a coin twice. The first flip comes up heads (E_1) and the second tails (E_2). Are these events independent?

$$P(E_1) = \frac{|\{(H,H),(H,T)\}|}{|\{(H,H),(H,T),(T,T),(T,H)\}|} = \frac{1}{2} \; (p(E_2) = \frac{1}{2}, \text{too}).$$

$$p(E_1 \cap E_2) = \frac{|\{(H,T)\}|}{4} = \frac{1}{4}$$

$$p(E_2 \mid E_1) = \frac{p(E_2 \cap E_1)}{p(E_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} = p(E_2) \quad \therefore \text{independent}$$

Independence of Events

Example:

Events: $E = \{2,4,6\},\ H = \{1,2,3\} \quad T = \{1,2,3,4\}$

Are E and H independent?

$$p(E) = \frac{1}{2}, \quad p(H) = \frac{1}{2}, \quad p(T) = \frac{2}{3}, \quad p(E \cap H) = \frac{1}{6}$$

$$p(E \mid H) = \frac{p(E \cap H)}{p(H)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \neq p(E)$$

$\therefore E$ and H are <u>not</u> independent!

Well ... why not??

(Continued …)

Independence of Events

$E = \{2,4,6\},\ \ H = \{1,2,3\} \quad T = \{1,2,3,4\}$

In our situation of $p(E \mid H)$, we are given that only 1, 2, or 3 was rolled. Only one of those shared with E, thus $p(E \mid H) = \frac{1}{3}$.

However, without the limitation of H, a roll will be a member of E without probability $\frac{1}{2}$.

This shows that the probability of E occurring changes if H is assumed to have occurred first. Thus, E is dependent on H, or E and H are not independent.

Note: E and T <u>are</u> independent:

$$p(E \cap T) = \frac{1}{3}, \text{ and so } p(E \mid T) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} = p(E)$$

Probabilistic Reasoning

Another approach to the door problem using conditional probability

3. Conditional Probability (Still assuming we pick door #1)

   Recall: $p(X \mid Y) = \dfrac{p(X \cap Y)}{p(Y)}$, or $P(Y) \cdot p(X \mid Y) = p(X \cap Y)$

   Let C_i be the event that the car is behind door i, and

   Let H_k be the event of the host opening door k

$$
\begin{aligned}
p(\text{win of switch}) &= p(H_3 \cap C_2) + p(H_2 \cap C_3) \\
&= p(C_2) \cdot p(H_3 \mid C_2) + p(C_3) \cdot p(H_2 \mid C_3) \\
&= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 \\
&= \frac{2}{3}
\end{aligned}
$$

(Note the application of the Addition Principle!)