Functions

2.3

Functions as Relations

- Consider $f(x) = x + 1, x \in \mathbb{Z}$
  - Alternate notation:
    - $f = \{(x, x+1) \mid x \in \mathbb{Z}\}$

Definition: *Function*

A function f from set X to Y, denoted $f : X \rightarrow Y$, is a relation from X to Y, where $f(x)$ is defined $\forall x \in X$ and, if $(x, y) \in f$, then y is the **only value** returned by $f(x)$.

Functions as Relations

Example:

$f = \{(x, x+1) \mid x \in \mathbb{Z}\}$

- **Is f a relation?**
- **Is $f(x)$ is defined for all integers?**
- **Is $x + 1$ is the only value returned by $f(x)$?**

Letter grades assigned to students:

- $X = \{$**Zeus, Leto, Apollo**$\}$
- $Y = \{A, B, C, D, E\}$
- $G = \{($**Zeus**$, A), ($**Leto**$, A), ($**Apollo**$, C)$

Playposit

Which of the relations below on $\{1,2,3,4\}$ are functions?

- $R = \{(1,3), (2,1), (3,2), (4,3)\}$
- $R = \{(1,1), (1,2), (2,3), (3,4), (4,1)\}$
- $R = \{(1,1), (2,1), (3,1), (4,1)\}$
- $R = \{(1,1), (2,2), (3,3)\}$

Function Terms

Let $f : X \to Y$ be a function and let $f(n) = p$.

- X is the ___**domain**___ of f
- Y is the ___**codomain**___ of f
- f ___**maps**___ X to Y
- p is the ___**image**___ of n
- n is the ___**pre-image**___ of p
- f's ___**range**___ is the set of all images of X's elements

Note: A function's range need note equal its codomain.

Function Terms

Example:

$g = \{(a, b) \mid b = a/2\},\ a \in \{0,2,4,8\},\ b \in \{0,1,2,3,4,5\}$

- Domain: $\{0,2,4,8\}$
- Codomain: $\{0,1,2,3,4,5\}$
- Image (of $(8,4)$): 4
- Pre-image (of $(8,4)$): 8
- Range (of g): $\{0,1,2,4\}$

Playposit

Given the function $f : A \to B$ where $f(x) = 2x + 1$, $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5,6,7,8,9\}$, fill in the blank with the term from the list below that corresponds to the set or value.

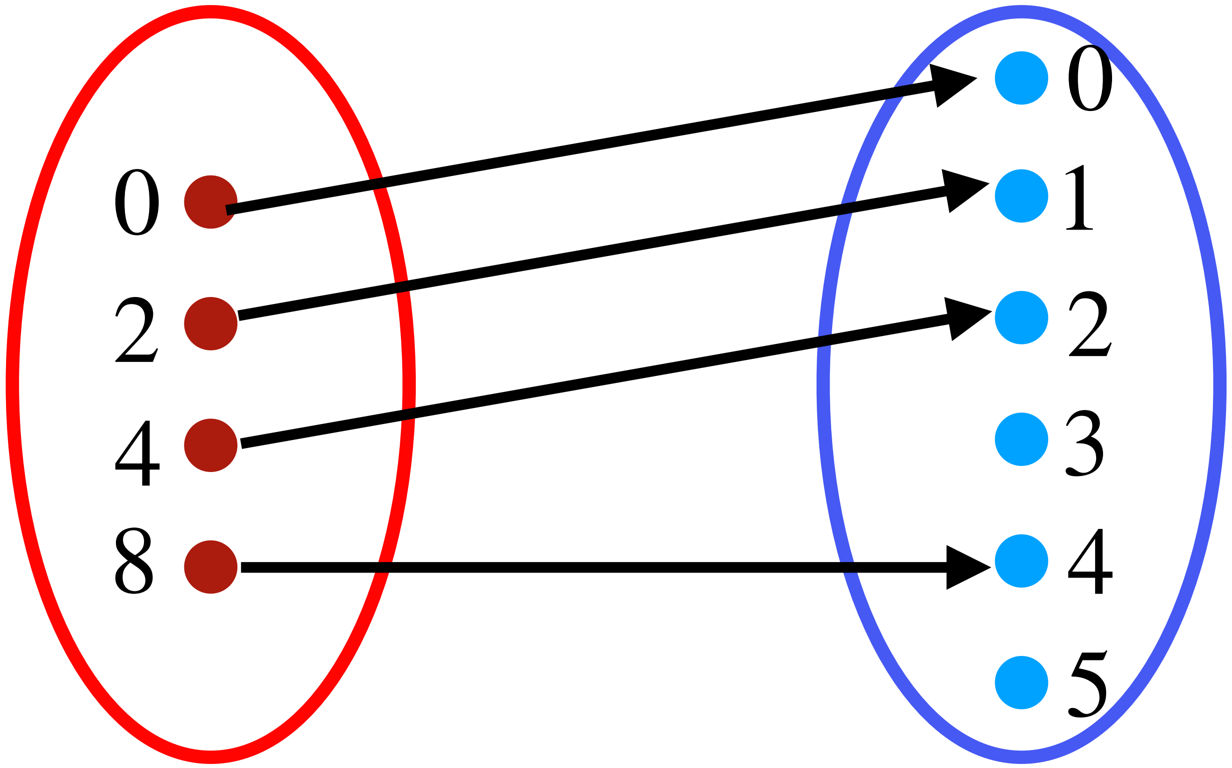
Terms: domain, codomain, image, pre-image, range

Sets/Values	Term
4 in the pair (4,9)	
{1,2,3,4,5,6,7,8,9}	
{1,2,3,4}	
{,3,5,7,9}	
9 in the pair (4,9)	

Digraph Representation

Example:

$$g = \{(a, b) \mid b = a/2\},\ a \in \{0,2,4,8\},\ b \in \{0,1,2,3,4,5\}$$



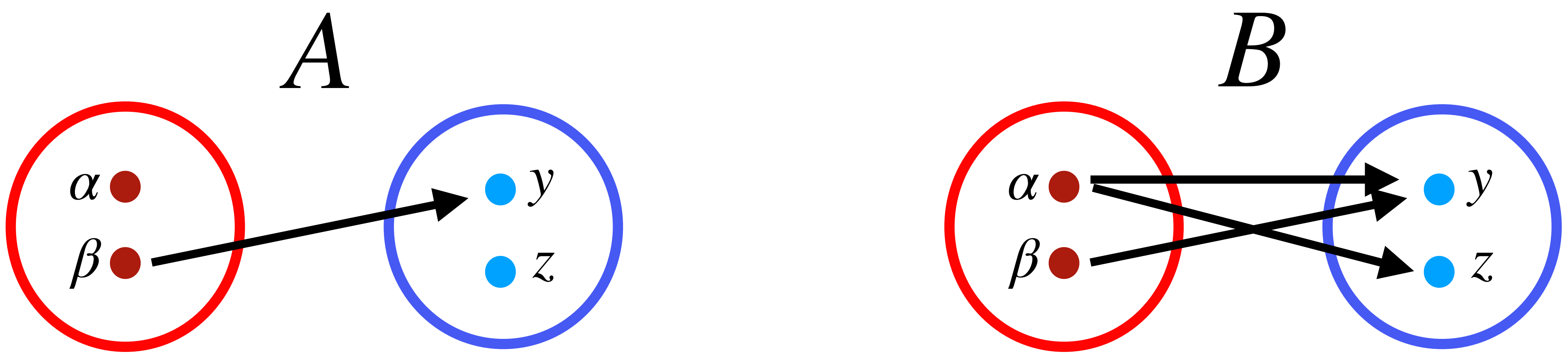
The incoming arrows identify the range members

Digraph Representation

Example:

$A = \{(\beta, y)\}$ from $\{\alpha, \beta\}$ to $\{y, z\}$

$B = \{(\alpha, y), (\alpha, z), (\beta, y)\}$ from $\{\alpha, \beta\}$ to $\{y, z\}$



Neither are functions:

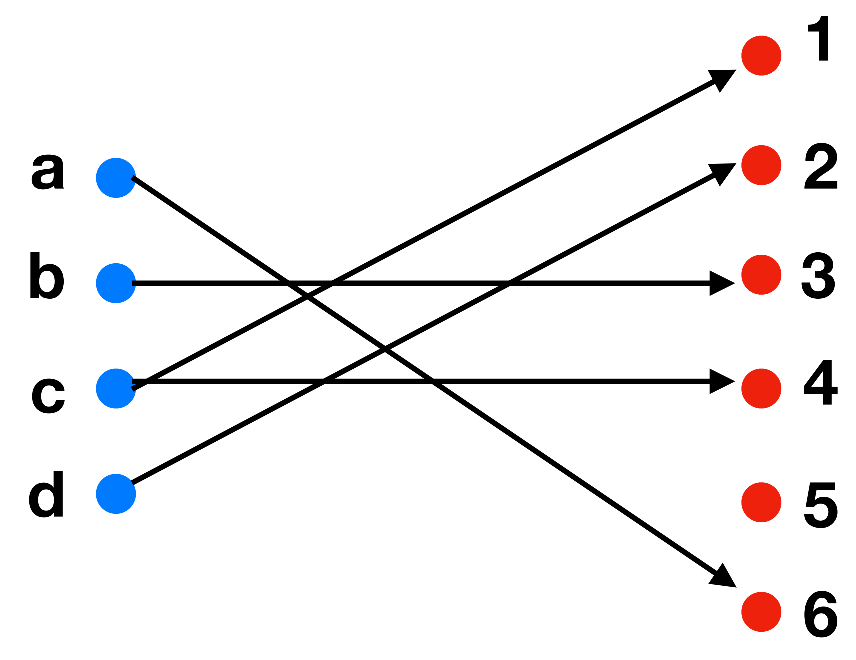
In A, α is unused (function not defined $\forall$ domain)

In B, α is related to multiple codomain values

Playposit

Let the relation R be a relation from A to B, where $A = \{a, b, c, d\}$ and $B = \mathbb{Z}$. The digraph below represents the relation. Is R a function?

A. Yes, it is a function.
B. No, it doesn't not map every element in the domain to the codomain
C. No, it maps some domain elements to multiple codomain elements
D. None of the above.



Two Functions You Need to Know

1. Floor ($\lfloor x \rfloor$)

Definition: *Floor Function*

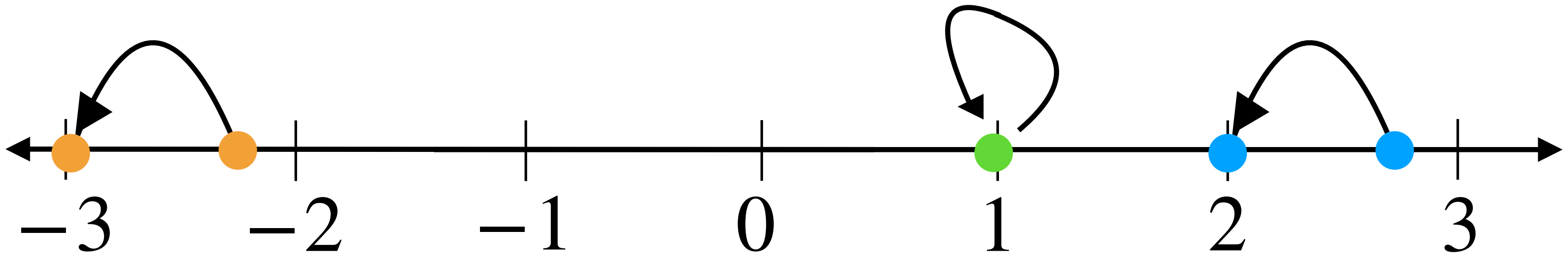
The floor of a real value n, denoted $\lfloor n \rfloor$, is the largest integer $\leq n$

Example:

$\lfloor 2.8 \rfloor = 2$

$\lfloor 1 \rfloor = 1$

$\lfloor -2.2 \rfloor = -3$



See also: `Math.floor()` in the Java API

Two Functions You Need to Know

1. Floor ($\lfloor x \rfloor$) (cont)

Using Floor for Rounding to the Nearest integer

Easy: Just add 0.5, then ‘floor it’. Example:

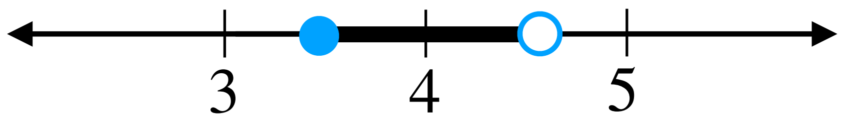
Round 3.50:

Round 3.99:

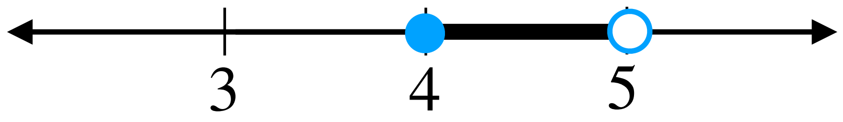
Round 4.49:

In General:

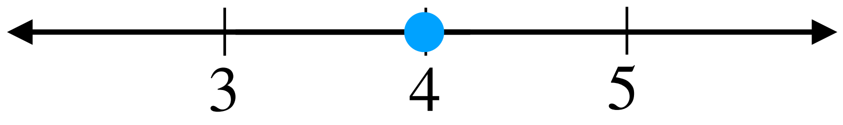
Range that rounds to 4



After 0.5 is added



After floor is applied



Playposit

For each value in the table, give the floor of the value in the corresponding blank.

x	$\lfloor x \rfloor$
4.5	
1	
7.7	
-3.4	

Two Functions You Need to Know

2. Ceiling ($\lceil x \rceil$)

Definition: *Ceiling Function*

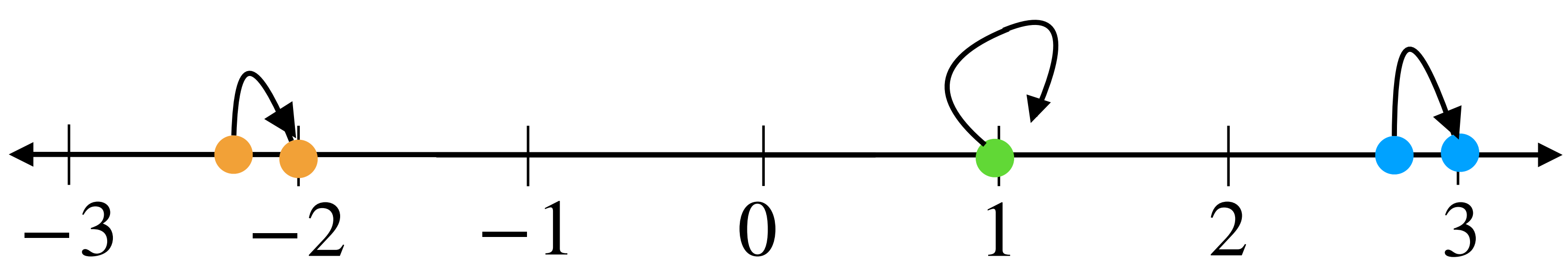
The ceiling of a real value m, denoted $\lceil m \rceil$, is the smallest integer $\geq m$

Example:

$\lceil 2.8 \rceil = 3$

$\lceil 1 \rceil = 1$

$\lceil -2.2 \rceil = -2$



Always move right on the number line

See also: `Math.ceil()` in the Java API

Two Functions You Need to Know

2. Ceiling ($\lceil x \rceil$)

Example:

Plan: \$0.50 for calls ≤ 10 min., plus \$0.05 per additional minute

Example: 11.5 minute call ⇒ 60 cents.

First try:

Cost (length) $= 50 + 5 \cdot \lceil \text{length} - 10 \rceil$ cents.

⇒But: Fails when length ≤ 10

Fixed:

$$\text{Cost (length)} = \begin{cases} 50 & \text{length} \leq 10 \\ 50 + 5 \cdot \lceil \text{length} - 10 \rceil & \text{Otherwise} \end{cases}$$

Playposit

For each value in the table, give the ceiling of the value in the corresponding blank.

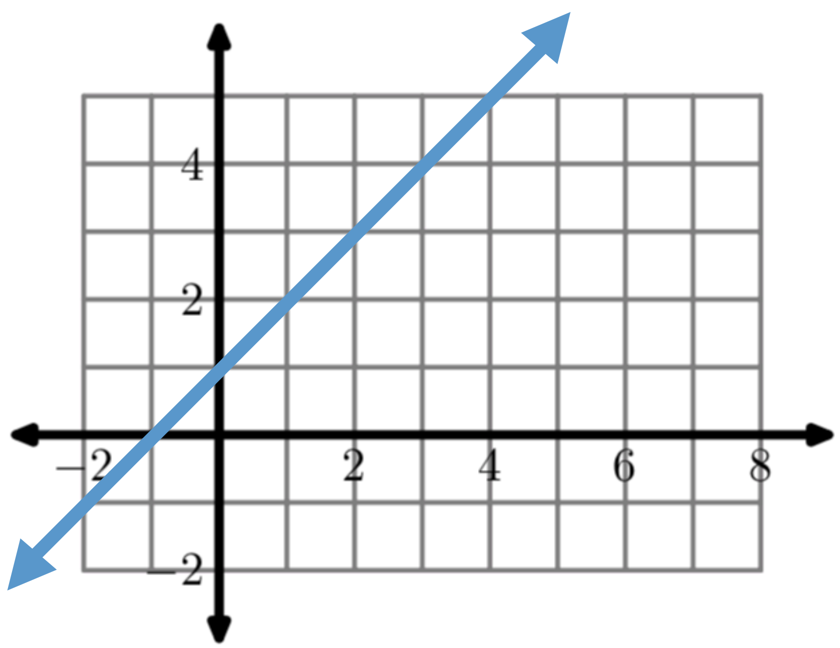
x	$\lceil x \rceil$
4.5	
1	
7.7	
-3.4	

Example: Type A UPC code Check Digits

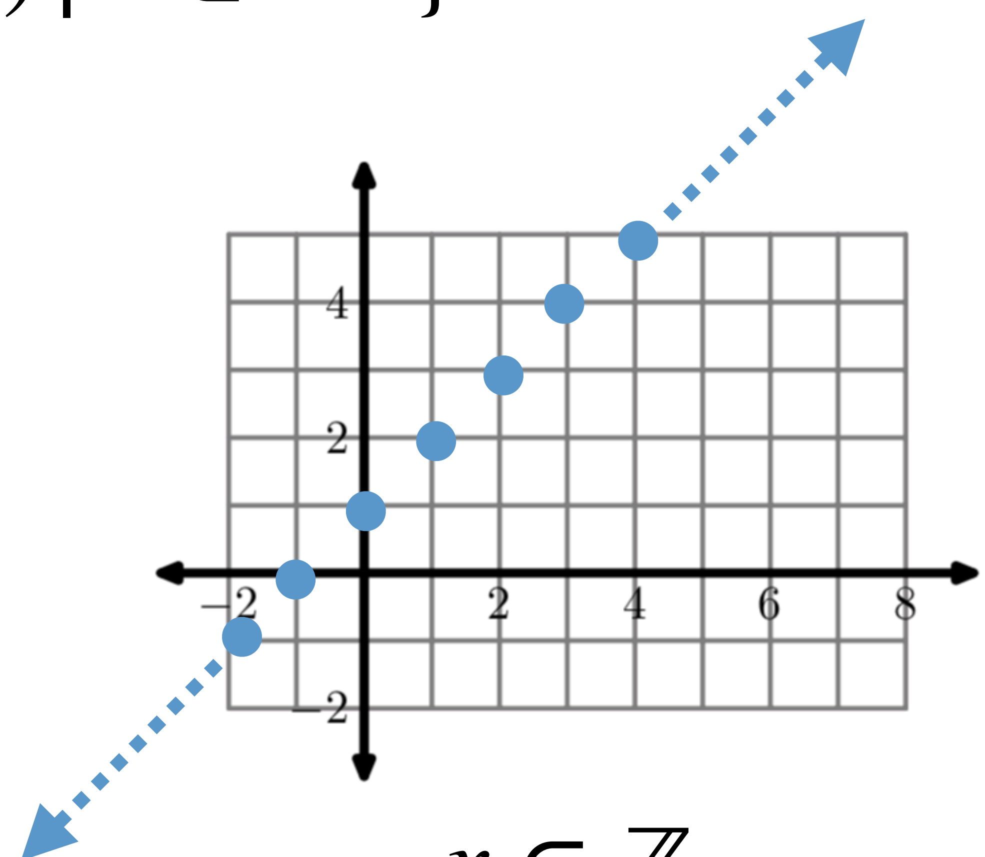
- The check digit equals the image of this function:
  - $s =$ Sum of digits in positions $1, 3, 5, 7, 9, \& \ 11$
  - $t =$ Sum of digits in positions $2, 4, 6, 8, \& \ 10$
  - $u = 3s + t$; the check digit is $(10 - u \,\%\, 10) \,\%\, 10$
- Using the above sample:
  - $s = 39$, $t = 24$, and $u = 3(39) + 24 = 141$
  - Check digit = $(10 - 141 \,\%\, 10) \,\%\, 10 = 9$

Plots of Functions

- Important Distinction: *Continuous* vs. *Discontinuous Functions*
- Consider: $f = \{(x, x+1) \mid x \in \dots\}$



$x \in \mathbb{R}$

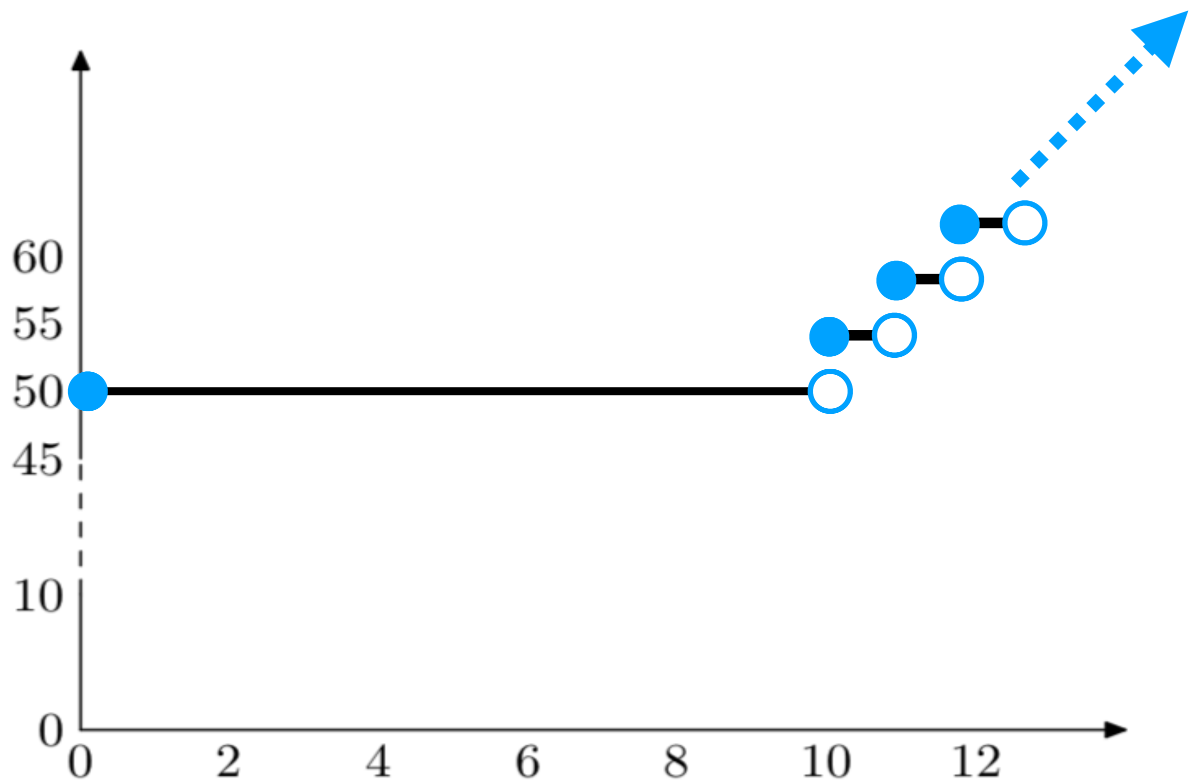


$x \in \mathbb{Z}$

Plots of Functions

- How should the plot of our long-distance calling plan function look?

$$\text{Cost (length)} = \begin{cases} 50 & \text{length} \leq 10 \\ 50 + 5 \cdot \lceil \text{length} - 10 \rceil & \text{Otherwise} \end{cases}$$



This is an example of a *piecewise function*

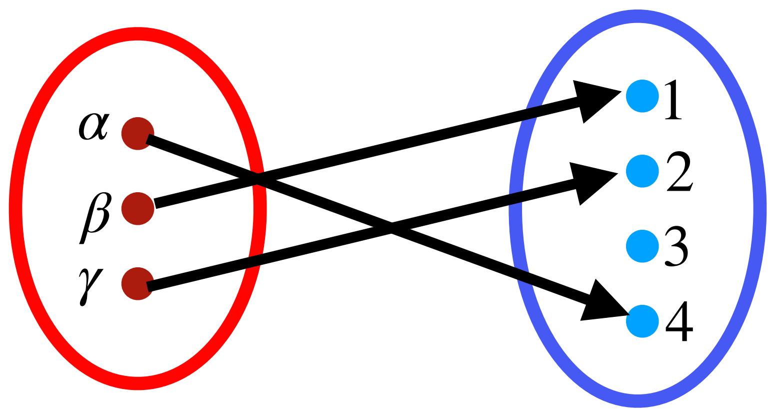
Categories of Functions: Injective

Definition: *Injective Functions* **(a.k.a One-to-One)**

A function f from set X to Y is injective if, for each $y \in Y, f(x) = y$ for at most one member of X

Example:

$F = \{(\alpha,4), (\beta,1), (\gamma,2)\}$ **from** $\{\alpha, \beta, \gamma\}$ **to** $\{1,2,3,4\}$



Each y has 0 or 1 incoming arrows

∴ it's injective

Categories of Functions: Surjective

Definition: *Surjective Functions* **(a.k.a Onto)**

A function f from set X to Y is surjective if, f's range is Y (that is, the range equals the codomain)

Example:

$F = \{(\alpha,4),(\beta,1),(\gamma,2)\}$ **from** $\{\alpha,\beta,\gamma\}$ **to** $\{1,2,3,4\}$

F **is <u>not</u> surjective: 3 is not used.**

Surjective functions: Each y **has** ≥ 1 **incoming arrows**

Categories of Functions: Bijective

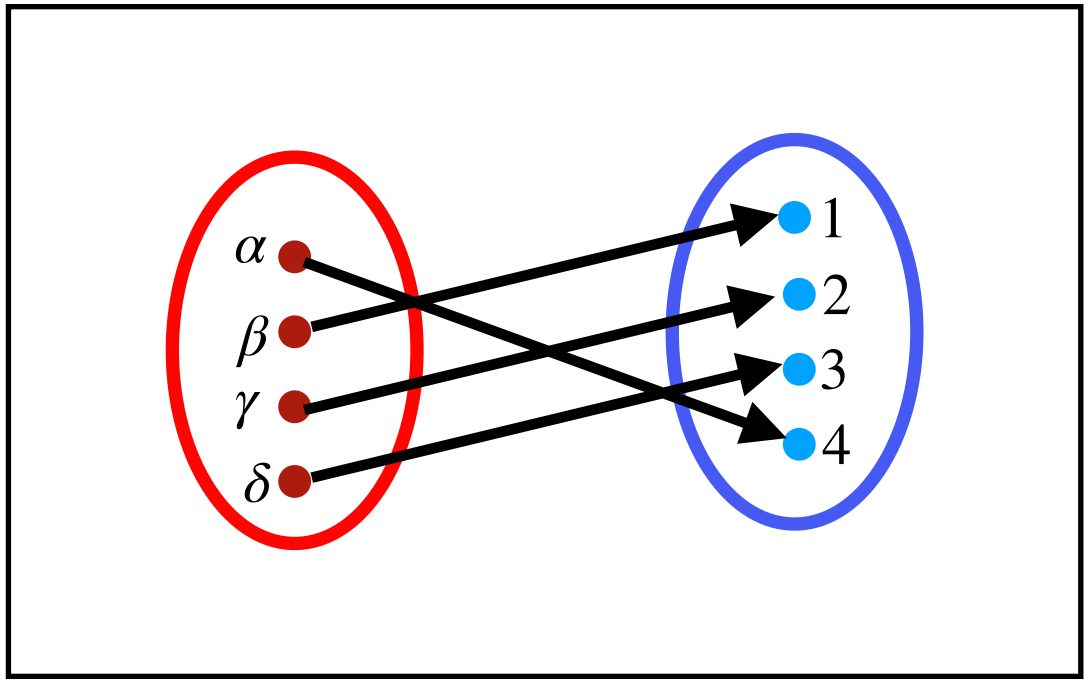
Definition: *Bijective Functions* **(a.k.a One-to-One Corresondence)**

A function f from set X to Y is bijective if it is both injective and surjective

Example:

Each y has exactly 1 incoming arrow

Note: $|X| = |Y|$



Playposit

Which of the following are true about the function below?

$f : A \to B,\ f(x) = |x|$ where

$A = \{-3, -2, -1, 0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$

- Injective
- Surjective
- Bijective

Odds and Ends

Definition: *Functional Composition*

Let $f : Y \to Z$ and $g : X \to Y$. The composition of f and g, denoted $f \circ g$, is the function $h = f(g(x))$, where $h : X \to Z$

Definition: *Inverse Functions*

The inverse of a bijective function f, denoted f^{-1}, is the relation $\{(y, x) \mid (x, y) \in f\}$

(Note the bijective requirement, otherwise the definition is the same as that of relational inverse)

Beyond Unary Functions

Definition: *Binary Function*

A binary function is a function $f : X \times Y \to Z$, ($f(x, y) = z$)

Example: Wind Chill (*)

$$WC(T, V) = (0.2175T - 35.75)V^{0.16} + 0.6215T + 35.74$$

where V is wind speed (mph) and T is air temp in ˚F

(Heat index is also a binary function, but messier!)

(*) Developed by the Joint Action Group for Temperature Indices and adopted by the US, UK, and Canada in Nov. 2001

Playposit

Which of the follow functions can be inverted?

All functions are from $A \to B$ where $A = \{-1,0,1,2\}$ and $B = \{0,1,2,3\}$

- $f(x) = |x|$
- $f(x) = x + 1$
- $f(x) = 0$
- $f(x) = 2 - x$