Proof by Induction

5.1

Review: Inductive and Deductive Reasoning

Definition: *Inductive Argument*

An argument that moves form specific observations to general conclusions

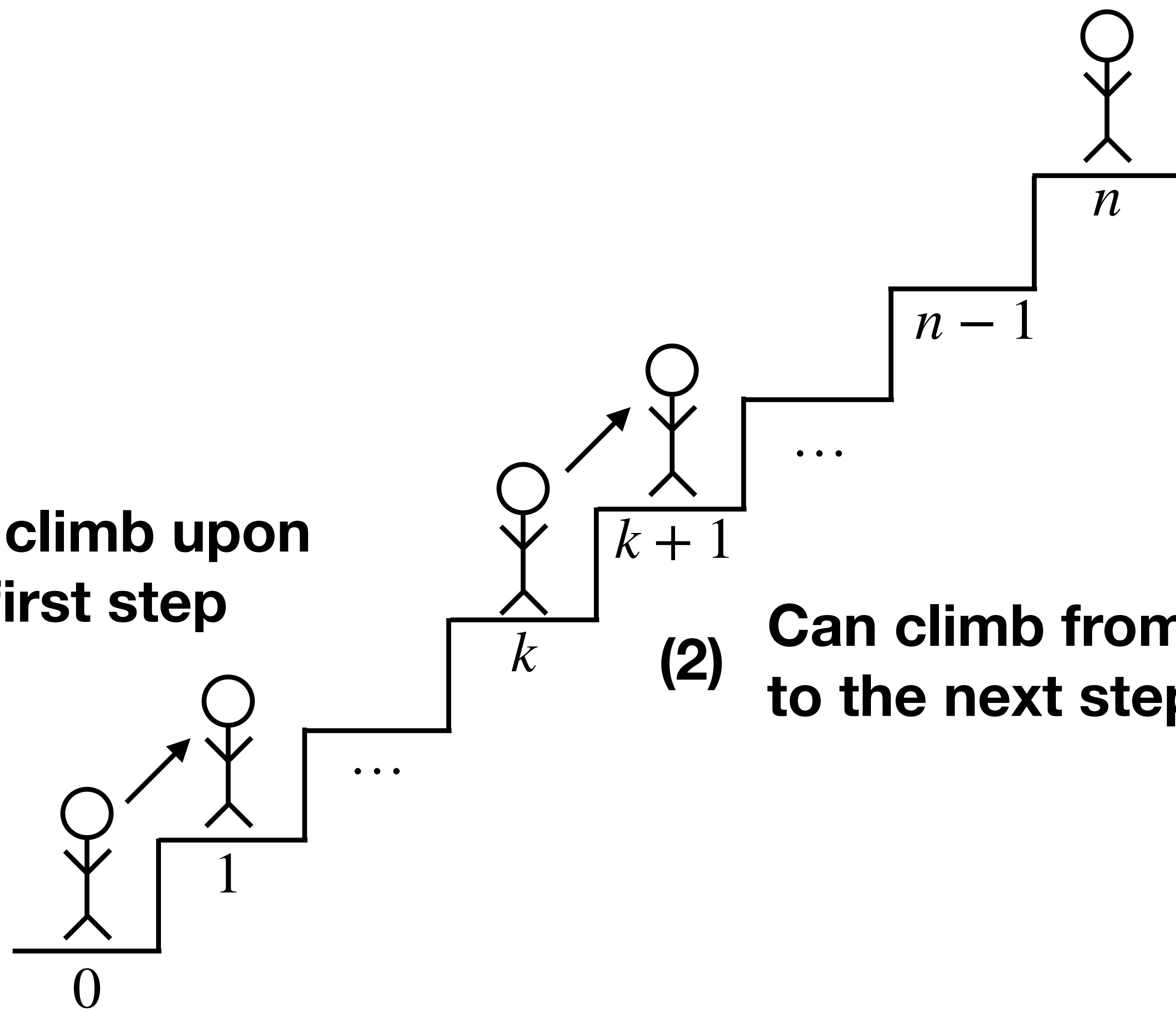
Definition: *Deductive Argument*

An argument that uses accepted general principles to explain a specific situation

Old Example: 3, 5, and 7 are prime numbers. Therefore all positive integers above 1 are prime numbers

The Big Idea

Example: Climbing Stairs

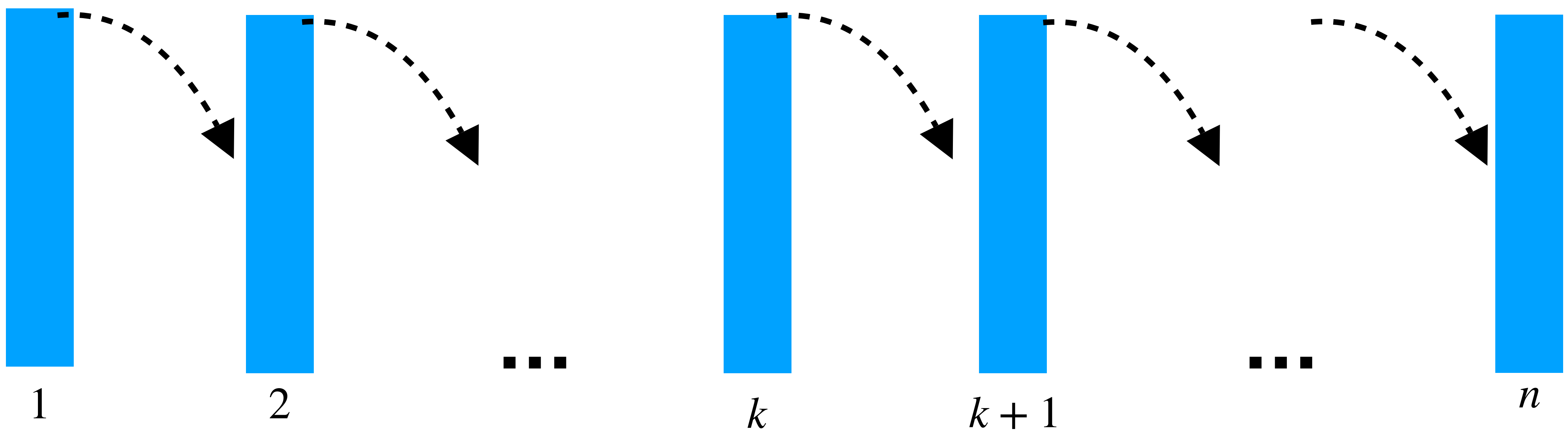


(1) **Can climb upon the first step**

(2) **Can climb from any step to the next step**

The Big Idea

Example: Dominoes



(1) 1st domino must fall onto the 2nd

(2) Domino k must fall onto domino $k+1$

But … How Does Induction *Prove* Anything?

After all, you could fall off the stairs, space dominoes too far apart, etc. ⇒ Integers are more predictable!

Consider $\mathbb{Z}^+$. If we can show that:

(a) A property holds for 1, and

(b) If the property holds for $n \in \mathbb{Z}$, then it also holds for the next integer $(n + 1)$,

Then we can conclude that the property will hold for 2 (b/c $1 \rightarrow 2$), 3 (b/c $2 \rightarrow 3$), etc. - that is, it will hold for all positive integers.

⇒ **This is the *First* Principle of Mathematical Induction**

The First Principle of Mathematical Induction

(often called weak induction)

Definition: *First Principle of Mathematical Induction*

If: (i) $P(a)$ is true for the starting point $a \in \mathbb{Z}^+$ and

(ii) (for any $k \in \mathbb{Z}^+ \; a \leq k$)

if $P(k)$ is true then $P(k+1)$ is true

Then: $P(n)$ is true for all $n \in \mathbb{Z}^+$, $n \geq a$.

(i) is the *Basis Step*, and (ii) is the *Inductive Step*

In the $p \to q$ form:

$$[P(a) \wedge \forall k \, (P(k) \to P(k+1))] \to \forall n \, P(n)$$

(Note that there's a proof within the proof!)

The Second Principle of Mathematical Induction

(often called strong induction)

Definition: *Second Principle of Mathematical Induction*

If: (i) $P(a)$ is true for the starting point $a \in \mathbb{Z}^+$ and

(ii) (for any $k \in \mathbb{Z}^+$, $a \leq k$)

If $P(j)$ is true for all $j \in \mathbb{Z}^+$ such that $a \leq j \leq k$

then $P(k+1)$ is true

Then: $P(n)$ is true for all $n \in \mathbb{Z}^+$, $n \geq a$.

Note the difference:

Strong $\Rightarrow$ All $P(i)$ from $P(a)$ through $P(k)$ are assumed true

Weak $\Rightarrow$ Just $P(k)$ is assumed true

But … Which One Should I Use?

- Whichever is appropriate! (if both are, use the easier!)
- Why? Because Weak Induction ≡ Strong Induction

(1) Easy to see that Strong → Weak

(2) Weak → Well-Ordering →Strong

(See Rosen 8/e Sec. 5.2 p. 361-2 and Ex. 41-43)

Definition: *The Well-Ordering Property:*

Every non-empty set of positive integers has a smallest member

Example #1: A summation

Conjecture: $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof (by Weak Induction):

Basis Step (base case):

Let $n = 1$. $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$, Ok!

Inductive Step:

If $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, then $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Inductive Hypothesis

(Continues…)

Example #1: A summation

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$ [By the Inductive Hypothesis]

$$= \frac{k^2 + k + 2k + 2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Thus, $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Example #2: An Inequality

Conjecture: $n^2 > n + 1,\ \forall n \geq 2,\ n \in \mathbb{Z}^+$

Proof (by Weak Induction):

Basis: Playposit: What is the base case?

A. $n = 1,\ 1^2 > 1 + 1$
B. $n = 0,\ 0^2 > 0 + 1$
C. $n = 2,\ 2^2 > 2 + 1$

Example #2: An Inequality

Conjecture: $n^2 > n + 1, \ \forall n \geq 2, \ n \in \mathbb{Z}^+$

Proof (by Weak Induction):

Basis: Let $n = 2$. $2^2 > 2 + 1$ (or $4 > 3$)? Yes!

Inductive Step: If $k^2 > k + 1$ then $(k+1)^2 > k + 2$

$$(k+1)^2 = k^2 + 2k + 1$$

By Inductive hypothesis, we know that $k^2 > k + 1$.

Thus $(k^2) + 2k + 1 > (k + 1) + 2k + 1 = 3k + 2$

(Continues …)

Example #2: An Inequality

We still must show that $3k + 2 \geq k + 2$. Let's simplify that.

By subtracting $k + 2$ from both sides gives $2k \geq 0$

Dividing both sides by 2 shows that $k \geq 0$ is all that needs to be shown

But we already know that $k \geq 0$ is true, because we were given that $k \geq 2$.

Thus, $(k+1)^2 > k + 2$

Therefore $k^2 > k + 1, \quad \forall k \geq 2$

Example #2: An Inequality

Here's an alternative inductive step, starting with the I.H.:

Inductive: If $k^2 > k + 1$, then $(k+1)^2 > k + 2$

(1)	$k^2 > k + 1$	[Given (Inductive Hypothesis)]
(2)	$k^2 + 2k + 1 > k + 1 + 2k + 1$	[Add $2k + 1$ to both sides]
(3)	$(k+1)^2 > 3k + 2$	[Algebra]
(4)	$k > 0$	[Given ($k \geq 2 \rightarrow k > 0$)]
(5)	$2k > 0$	[Multiply both sides by 2]
(6)	$3k + 2 > k + 2$	[Add $k + 2$ to both sides]
(7)	$(k+1)^2 > k + 2$	[(3),(6),Transitivity]

Problem: Very 'magical': Hard to see that adding $2k + 1$ and assuming $k > 0$ are the right things to do!

The Basis Step Only *Seems* Pointless

Conjecture: $\sum_{i=0}^{n-1} 2^i = 2^n, \ \forall n \geq 1$

Proof (by Weak Induction):

Basis: $n = 1, \quad \sum_{i=0}^{0} 2^i = 2^1$ (or $2^0 = 2^1$)

But: The inductive step still works!

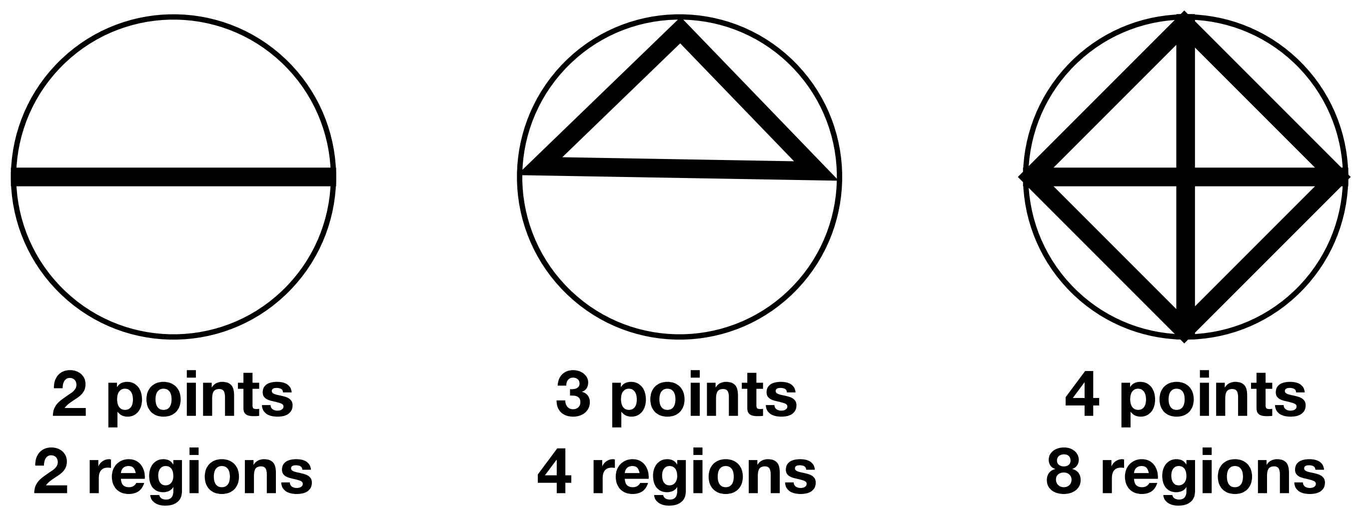
Inductive Step: if $\sum_{i=0}^{k-1} 2^i = 2^k$ then $\sum_{i=0}^{k} 2^i = 2^{k+1}$

$$\sum_{i=0}^{k} 2^i = \sum_{i=0}^{k-1} 2^i + 2^k = 2^k + 2^k \text{ [by the I.H] } = 2(2^k) = 2^{k+1}$$

What is provable: $\sum_{i=0}^{k-1} 2^i = 2^k - 1$

Some “Obvious” Patterns That Aren’t

1. Circle Division by Chords



p	regions
2	2
3	4
4	8
$\vdots$	$\vdots$
p	2^{p-1}?

No! $p = 6$ has just 31 regions

The correct expression:

$$\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

$$= 1 + \binom{n}{2} + \binom{n}{4}$$

(Note: No three chords may intersect at the same point.)

Some “Obvious” Patterns That Aren’t

2. Is $n^2 - n + 41$ prime $\forall n \geq 1$?

No! It must fail at $n = 41$ ($41^2 - 41 + 41$ is clearly composite). However, all members of the sequence $\{n^2 - n + 41\}_{n=1}^{40}$ are prime.

3. Is $991n^2 + 1$ never a perfect square $\forall n \geq 1$?

The 1st n that makes $991n^2 + 1$ a perfect square is

12,055,735,790,331,447,442,538,767

Strong Induction

- Review:
  - In *Strong Induction,* $P(a)$ must be true and, for any $k \geq a$, <u>if</u> $P(a) \wedge P(a+1) \wedge \ldots \wedge P(k-1) \wedge P(k)$ is true, then $P(k+1)$ is true.

<u>Weak</u> needs only the truth of the preceding case to show the truth of the next case

<u>Strong</u> needs the truth of multiple preceding cases.
$\Rightarrow$ Only use those preceding cases that are required by your situation!

Example: Strong Induction

Conjecture: Let $a_0 = 1, a_1 = 2$, and $a_2 = 3$. Also assume $a_k = a_{k-1} + a_{k-2} + a_{k-3}$. Is $a_n \leq 2^n, \forall n \geq 3$?

Proof (by Strong Induction):

Basis: We start w/ $n = 3$ b/c $n = 0,1,2$ are given.

$a_3 = a_2 + a_1 + a_0 = 3 + 2 + 1 = 6 \leq 2^3 = 8$. Ok!

Inductive Step: if $a_i \leq 2^i, \forall i \;\; 3 \leq i < k$, then $a_k \leq 2^k$.

Begin with $a_k = a_{k-1} + a_{k-2} + a_{k-3}$

(continues…)

(Note, that the IH goes to k-1 instead of k.)

Example: Strong Induction

By the Inductive Hypothesis we know that
$a_{k-1} \leq 2^{k-1}$, $a_{k-2} \leq 2^{k-2}$, and $a_{k-3} \leq 2^{k-3}$.

It follows that $a_k \leq 2^{k-1} + 2^{k-2} + 2^{k-3}$.

In turn, this is $\leq 2^{k-1} + 2^{k-2} + \ldots + 2^0$.

In a prior example, we claimed that is the sum equal to $2^k - 1$.

Combining these expressions, we see that $a_k \leq 2^k - 1$ which means that $a_k \leq 2^k$

Therefore, $a_n \leq 2^n, \ \forall n \geq 3$

Structural Induction

- Simple Idea: Apply induction (weak or strong) to "structures", such as:
  - Strings
  - Binary trees
  - Program statements

    ⋮

Ex: In a complete binary tree,

leaf nodes = # internal nodes + 1