Proof Examples

Example 1

Conjecture: Every odd integer is the difference of two squares.

Proof (Direct): Let n be an odd integer. $\exists_{k \in \mathbb{Z}}$ s.t. $n = 2k + 1$

To gain some insight:

$3 = 2(1) + 1 = 2^2 - 1^2$, $5 = 2(4) + 1 = 3^2 - 2^2$,

$7 = 2(3) + 1 = 4^2 - 3^2$, $27 = 2(13) + 1 = 14^2 - 13^2$

Observation 1: odd numbers seem to be the difference of two consecutive squares.

Observation 2: For an odd number, $n = 2k + 1$, it seems to be the sum of the squares $(k+1)^2 - k^2$.

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$3 = 2(1) + 1 = 2^2 - 1^2,\ 5 = 2(4) + 1 = 3^2 - 2^2,$

WARNING: We have not proved this yet!

$7 = 2(3) + 1 = 4^2 - 3^2,\ 27 = 2(13) + 1 = 14^2 - 13^2$

Observation 1: odd numbers seem to be the difference of two consecutive squares.

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Example 1

Conjecture: Every odd integer is the difference of two squares.

Proof (Direct): Let n be an odd integer.

$\exists_{k \in \mathbb{Z}}$ s.t. $n = 2k + 1$

Observe: $(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1$

So, when we have $n = 2k + 1$, we will add and subtract k^2 to the right side: $n = 2k + 1 + k^2 - k^2$

Which can be factored to: $n = (k+1)^2 - k^2$.

Therefore, every odd integer is the difference of two squares.

Example 2(a)

Conjecture: If x is irrational, then $\frac{1}{x}$ is irrational.

Proof (Contrapositive): Assume $\frac{1}{x}$ is rational.

By definition, $\exists_{p,q \in \mathbb{Z}}$ s.t. $\frac{1}{x} = \frac{p}{q}$. $p \neq 0$ because $\frac{1}{x} \neq 0$

Solving for x, we get $x = \frac{q}{p}$. Since p and q are both integers, x is rational.

Thus we have shown that the contrapositive is true.

Therefore, if x is irrational, then $\frac{1}{x}$ is irrational.

Example 2(b)

Conjecture: If x is irrational, then $\frac{1}{x}$ is irrational.

Proof (Contradiction): Assume x is irrational but $\frac{1}{x}$ is rational.

By definition, $\exists_{p,q\in\mathbb{Z}}$ s.t. $\frac{1}{x} = \frac{p}{q}$. $p \neq 0$ because $\frac{1}{x} \neq 0$

Solving for x we get $x = \frac{p}{q}$. We know p and q are integers so this makes x rational, which is a contradiction.

Therefore, If x is irrational, then $\frac{1}{x}$ is irrational.

Example 3

Conjecture: Pick a list of 22 days in a year. At least four of those days fall on the same day of the week.

Proof (Contradiction): Assume not. Assume that in our list of 22 days that no day of the week occurs more than 3 times.

Let d_i be the number of times i^{th} day of the week occurs, where $1 \leq i \leq 7$.

If $d_i \leq 3$, then $\sum_{i=1}^{7} d_i \leq \sum_{i=1}^{7} 3 \leq 7 * 3 = 21$. This is a contradiction because we assumed our list had 22 days on it.

Therefore, if we pick a list of 22 days in a year, then at least four of those days fall on the same day of the week.

Example 4

Conjecture: If $x, y \in \mathbb{R}$, then $|x| + |y| \geq |x + y|$

Proof (By Cases): Assume x and y are real numbers.

We know that if $x \geq 0,\ |x| = x$ and if
$x < 0,\ |x| = -x$

Case 1: $x \geq 0,\ y \geq 0$.

$|x| + |y| = x + y$.

$|x + y| = x + y$

Thus $|x| + |y| \geq |x + y|$

Example 4

Conjecture: If $x, y \in \mathbb{R}$, then $|x| + |y| \geq |x + y|$

Proof (By Cases): Assume x and y are real numbers.

We know that if $x \geq 0$, $|x| = x$ and if $x < 0$, $|x| = -x$

Case 2: $x \geq 0,\ y < 0,\ x + y \geq 0.$

$|x| + |y| = x + (-y) = x - y.$

$-y \geq y$, so $x + -y \geq x + y.$

Since $x + y \geq 0$, $|x + y| = x + y.$

Thus $|x| + |y| = x + -y \geq x + y = |x + y|$

Example 5

Conjecture: If $x, y \in \mathbb{R}$, then $|x| + |y| \geq |x + y|$

Proof (By Cases): Assume x and y are real numbers.

We know that if $x \geq 0$, $|x| = x$ and if $x < 0$, $|x| = -x$

Case 3: $x \geq 0$, $y < 0$, $x + y < 0$.
$|x| + |y| = x + (-y) = x - y$.

Since $x + y < 0$, $|x + y| = -(x + y) = -x - y$.

$x \geq -x$ so $x - y \geq -x - y$

Thus $|x| + |y| = x - y \geq -x - y = |x + y|$

Example 5

Conjecture: If $x, y \in \mathbb{R}$, then $|x| + |y| \geq |x + y|$

Proof (By Cases): Assume x and y are real numbers.

Case 4 & 5: Same as case 2 & 3 with x and y flipped

Case 6: $x < 0, \ y < 0$.

$|x| + |y| = -x - y$.

$|x + y| = -(x + y) = -x - y$.

Thus, $|x| + |y| \geq |x + y|$

Therefore, $|x| + |y| \geq |x + y|$ in all cases.

Example 6

Conjecture: The following three statements about $x \in \mathbb{R}$ are equivalent: (i) x is rational, (ii) $x/2$ is rational, and (iii) $3x - 1$ is rational.

Proof: To show that these three are equivalent, it is sufficient to show (i)→(ii), (ii)→(iii), and (iii)→(i).

Why is that sufficient?

1. (i)→(iii) ≡ (i)→(ii)∧ (ii)→(iii)
2. (ii)→(i) ≡ (ii)→(iii)∧ (iii)→(i)
3. (iii)→(ii) ≡ (iii)→(i)∧ (i)→(ii)

Example 6

Conjecture: The following three statements about $x \in \mathbb{R}$ are equivalent: (i) x is rational, (ii) $x/2$ is rational, and (iii) $3x - 1$ is rational.

Proof: (i)→(ii) (direct): Assume x is rational.

By definition, $\exists_{p,q \in \mathbb{Z}}$ s.t. $x = \dfrac{p}{q}$.

$x/2 = \dfrac{\frac{p}{q}}{2} = \dfrac{p}{2q}$. $2q$ is an integer, therefore x/2 is rational.

Therefore, (i)→(ii).

(continued)

Example 6

Conjecture: The following three statements about $x \in \mathbb{R}$ are equivalent: (i) x is rational, (ii) $x/2$ is rational, and (iii) $3x - 1$ is rational.

Proof: (ii)→(iii) (direct): Assume $x/2$ is rational.

By definition, $\exists_{p,q \in \mathbb{Z}}$ s.t. $x/2 = \dfrac{p}{q}$.

$$x = \frac{2p}{q},\ \ 3x - 1 = 3(\frac{2p}{q}) - 1 = \frac{6p}{q} - 1 = \frac{6p}{q} - \frac{q}{q} = \frac{6p - q}{q}$$

$6p - q \in \mathbb{Z}$, so $3x - 1$ is rational.

Therefore, (ii)→(iii). (continued)

Example 6

Conjecture: The following three statements about $x \in \mathbb{R}$ are equivalent: (i) x is rational, (ii) $x/2$ is rational, and (iii) $3x - 1$ is rational.

Proof: (iii)→(i) (direct): Assume $3x - 1$ is rational.

By definition, $\exists_{p,q \in \mathbb{Z}}$ s.t. $3x - 1 = \dfrac{p}{q}$.

$$3x = \frac{p}{q} + 1 = \frac{p+q}{q}. \quad x = \frac{\frac{p+q}{q}}{3} = \frac{p+q}{3q}.$$

$p + q \in \mathbb{Z}, 3q \in \mathbb{Z}$ so x is rational.

Therefore, (iii)→(i).

Thus, the statements x is rational, $x/2$ is rational, and $3x - 1$ is rational are equivalent.