Quantification

From last time:

Definition: *Proposition*

A declarative sentence that is either true (**T**) or false (**F**), but not both.

- Tucson summers never get above 100 degrees
- If the doorbell rings, then my dog will bark.
- You can take the flight if and only if you bought a ticket
- But **NOT** $x < 10$

Why do we want use variables?

- Propositional logic is not always expressive enough
- Consider: “All students like summer vacation”
- Should be able to conclude that “If Joe is a student, he likes summer vacation”.
- Similarly, “If Rachel is a student, she likes summer vacation” and so on.
- Propositional Logic does not support this!

Predicates

Definition: *Predicate (a.k.a. Propositional Function)*

A statement that includes at least one variable and will evaluate to either **true** or **false** when the variables(s) are assigned value(s).

- Example:

$$S(x) : (-10 < x) \wedge (x < 10)$$

$$E(a, b) : a \textbf{ eats } b$$

- These are not complete!

Predicates

Definition: *Domain (a.ka. Universe) of Discourse*

The collection of values from which a variable's value is drawn.

- Example:

$$S(x) : (-10 < x) \wedge (x < 10),\ \ x \in \mathbb{Z}$$

$$E(a, b) : a \textbf{ eats } b,\ a \in \textbf{People},\ b \in \textbf{Vegetables}$$

- In This Class: Domains may **NOT** hide operators
  - OK: **Vegetables**
  - Not-OK: **Raw Vegetables** **(Vegetable ∧ ¬Cooked)**

Playposit Question

Which of the following domains contain hidden operators?

- Clean clothes
- People
- Open windows
- UA students
- Countries

Evaluating Predicates

$$S(x) : (-10 < x) \wedge (x < 10),\ \ x \in \mathbb{Z}$$

$E(a, b) : a$ **eats** $b,\ a \in$ **People**, $b \in$ **Vegetables**

- Can evaluate predicates at specific values (making them propositions):
  - What is $E(Joe, Asparagus)$?
  - What is $S(0)$?

Combining Predicates with Logical Operators

- In $E(a, b)$: a eats b, $a \in$ people, $b \in$ vegetables, change the domain of b to “raw vegetables”.
  - $E(a, b)$: a eats b, $a \in$ people, $b \in$ vegetables
  - $C(b)$: b is cooked, $b \in$ vegetables
- Combining the two:
  - $E(a, b) \wedge \neg C(b)$, $a \in$ people, $b \in$ vegetables

Quantification

- Idea: Establish truth of predicates over sets of values.
- Two common generalizations:
    - *Universal Quantification* ($\forall x P(x), x \in D$)
        - Considers all values from the domain of discourse
    - *Existential Quantification* ($\exists x\, S(x), x \in D$)
        - Considers one or more values form the domain of discourse

Note: Do not use the books non-standard $\exists ! x$ notation
("uniqueness quantifier", Rosen 8/e p.46)

Evaluating Quantified Predicates

- *Universal Quantification:*

  Universal quantification of $P(x)$, $\forall x\, P(x)$, is the statement "$P(x)$ holds for all objects x in the domain of discourse"

  $\forall x\, P(x)$ is true only when $P(x)$ is true for **every** x in the domain, and false otherwise.

- Example 1:
  - $Q(x) : x = x^2, x \in \mathbb{R}$
  - $\forall x\, Q(x), x \in \{-1,0,1\}$?

  $Q(-1) =$ **False**,
  $-1 \neq (-1)^2$

- A value x for which $P(x)$ is false is a ***counterexample*** $\forall x\, P(x)$

Evaluating Quantified Predicates

- *Universal Quantification:*

  Universal quantification of $P(x)$, $\forall x\, P(x)$, is the statement "$P(x)$ holds for all objects x in the domain of discourse"

  $\forall x\, P(x)$ is true only when $P(x)$ is true for **every** x in the domain, and false otherwise.

- Example 2:
  - $P(x, y) : x + y$ is even, $x, y \in \mathbb{Z}$
  - $\forall x \forall y\ P(x, y),\ x, y \in \mathbb{Z}^{odd}$ **True! (Easy to prove)**

Evaluating Quantified Predicates

- *Existential Quantification:*

  Existential quantification of $P(x)$, $\exists x\, P(x)$, is "There exists an element x in the domain of discourse such that $P(x)$"

  $\exists x\, P(x)$ is true if **at least one** element x in the domain such that $P(x)$ is true

- Example 1:
  - $Q(x) : x = x^2, x \in \mathbb{R}$
  - $\exists x\, Q(x), x \in \{-1,0,1\}$?

$Q(0) =$ **True**,
$0 \neq 0^2$

Evaluating Quantified Predicates

- *Existential Quantification:*

  Existential quantification of $P(x)$, $\exists x\, P(x)$, is “There exists an element x in the domain of discourse such that $P(x)$”

  $\exists x\, P(x)$ is true if **at least one** element x in the domain such that $P(x)$ is true

- Example 2:
  - $P(x, y) : x + y$ is even, $x, y \in \mathbb{Z}$
  - $\exists x \exists y\; P(x, y),\; x, y \in \mathbb{Z}^{odd}$

True! Universal quantifier covers existential

Quantifications in Propositional Logic

- *Universal Quantification*
  - $\forall x P(x), x \in D \equiv P(d_0) \wedge P(d_1) \wedge P(d_2) \wedge \ldots$

- *Existential Quantification*
  - $\exists x\, S(x), x \in D \equiv S(d_0) \vee S(d_1) \vee S(d_2) \vee \ldots$

Converting From Quantified Predicates to Propositional Logic

Example 1:

Let $P(x)$: x is a prime number, $x \in \mathbb{Z}$

Express $\forall x\, P(x)$, $x \in \{3,5,7,9,11\}$ in propositional logic.

$$P(3) \wedge P(5) \wedge P(7) \wedge P(9) \wedge P(11)$$

What is its truth value?

Converting From Quantified Predicates to Propositional Logic

Example 2:

Let $P(x)$: x is a prime number, $x \in \mathbb{Z}$

Express $\exists x\, P(x)$, $x \in \{3,5,7,9,11\}$ in propositional logic.

$$P(3) \vee P(5) \vee P(7) \vee P(9) \vee P(11)$$

What is its truth value?

Playposit Question

Let $P(x)$ be the predicate "x is a prime number", where $x \in \mathbb{Z}$. Which of the following quantified statements are true?

- $\forall x P(x),\ x \in \{3,5,7,13,17\}$
- $\forall P(x),\ x \in \mathbb{Z}^+$
- $\exists x P(x),\ x \in \mathbb{Z}^+$
- $\exists x P(x),\ x \in \{2,3,4,5,6,7,8,9\}$
- $\forall x \neg P(x),\ x \in \mathbb{Z}^+$
- $\exists x \neg P(x),\ x \in \mathbb{Z}^+$

Examples: Converting from English to Quantified Predicates

Example: Universal Quantification

- Consider this conversational English statement:

  All of Nichole's adult friends are computer scientists.

- How can we express that statement in logic notation?

— —WARNING — —

Several **INCORRECT** versions follow…

Only the last version is correct!

Example: Universal Quantification

- Consider this conversational English statement:

All of Nichole's adult friends are computer scientists.

- How can we express that statement in logic notation?

Let $C(x)$: x is a computer scientist, $x \in$ **People**

$$\forall x \ T(x), x \in \textbf{People}$$

Stilted English: For every person x, x is a computer scientist.

Conversationally: All people are computer scientists

PROBLEM: This is not quite the desired meaning

IDEA: Let's focus the domain!

Example: Universal Quantification

- Attempt #2: All of Nichole's adult friends are computer scientists.

Let $C(x)$: x is a computer scientist, $x \in$ **Nichole's adult friends**

$$\forall x \ T(x), x \in \textbf{Nichole's adult friends}$$

Stilted English: For each of Nichole's adult friends x, x is a computer scientist.

Conversationally: All of Nichole's adult friends are computer scientists.

PROBLEM: The domain has a hidden predicate

IDEA: Let's create a new predicate.

Example: Universal Quantification

- Attempt #3: All of Nichole's adult friends are computer scientists.

Let $C(x)$: x is a computer scientist, $x \in$ **People**

Let $F(x)$: x is Nichole's adult friend, $x \in$ **People**

$\forall x \ (C(x) \wedge F(x)), x \in$ **People**

Stilted English: For every person x, x is an adult computer scientist and Nichole's friend.

Conversationally: All people are adult computer scientists and Nichole's friend.

PROBLEM: If true, implies that *all people* are Nichole's friends!

IDEA: Try a different compound predicate

Example: Universal Quantification

- Attempt #4: All of Nichole's adult friends are computer scientists.

Let $C(x)$: x is a computer scientist, $x \in$ **People**

Let $F(x)$: x is Nichole's adult friend, $x \in$ **People**

$\forall x\ (F(x) \rightarrow C(x)), x \in$ **People**

Stilted English: For every person x, if x is Nichole's friend, then x is a computer scientist.

Conversationally: All of Nichole's adult friends are computer scientists.

PROBLEM: Isn't $F(x)$, really two predicates in one?

IDEA: Break it apart

[Why not $C(x) \rightarrow F(x)$? That says all computer scientists are Nichole's adult friends]

Example: Quantification

- Attempt #5: All of Nichole's adult friends are computer scientists.

Let $C(x)$: x is a computer scientist, $x \in$ **People**

Let $F(x)$: x is Nichole's friend, $x \in$ **People**

Let $A(x)$: x is an adult, $x \in$ **People**

$\forall x \ ((A(x) \wedge F(x)) \rightarrow C(x)), x \in$ **People**

Stilted English: For every person x, if x is an adult and is Nichole's friend, then x is a computer scientist.

Conversationally: All of Nichole's adult friends are computer scientists.

— —SUCCESS! — —

(This is the version to learn!)

Playposit Question

Which of the below correctly translates the following sentence into logic: “All binary digits are either 0 or 1”

Let $P(x)$: x is a 0, $x \in$ Digits

$Q(x)$: x is a 1, $x \in$ Digits

$B(x)$: x is binary, $x \in$ Digits

A. $\forall x(B(x) \wedge (P(x) \vee Q(x)))$, $x \in$ Digits
B. $\forall x(P(x) \vee Q(x))$, $x \in$ Binary Digits
C. $\forall x(P(x) \oplus Q(x))$, $x \in$ Binary Digits
D. $\forall x(B(x) \wedge (P(x) \oplus Q(x)))$, $x \in$ Digits
E. $\forall x(B(x) \rightarrow (P(x) \vee Q(x)))$, $x \in$ Digits
F. $\forall x(B(x) \rightarrow (P(x) \oplus Q(x)))$, $x \in$ Digits

Implicit Quantification

- The “all” can be implicit in the English statement
- Example:
  - Adding an odd # to itself produces an even #

$O(x)$: x **is odd**, $x \in \mathbb{R}$

$E(x)$: x **is even**, $x \in \mathbb{R}$

$$\forall x \ (O(x) \rightarrow E(x + x)), \ x \in \mathbb{Z}$$

$$\forall x \ (O(x) \rightarrow \overline{O(x + x)}), \ x \in \mathbb{Z}$$

Note the implicit $\forall$ is implicit in the sentence

Example: Existential Quantification

- Consider this conversational English statement:

At least one breed of dog is cute.

- How can we express that statement in logic notation?

Let $C(x)$: x **is cute**, $x \in$ **Dog Breeds**

$\exists x \; C(x)$, $x \in$ **Dog Breeds**

English: There is at least one dog breed x such that x is cute.

Example: Existential Quantification

- Express this more specific statement in logic:

Some of the large fluffy dog breeds are cute.

Let $L(x)$: x **is large**, $x \in$ **Dog Breeds**

Let $F(x)$: x **is fluffy**, $x \in$ **Dog Breeds**

Let $C(x)$: x **is cute**, $x \in$ **Dog Breeds**

$\exists x\,(L(x) \wedge F(x) \wedge C(x))$, $x \in$ **Dog Breeds**

These alternatives don't work! Why?

$\exists x\,(L(x) \wedge F(x)) \rightarrow C(x)$, $x \in$ **Dog Breeds**

$\exists x\,(L(x) \wedge C(x)) \rightarrow F(x)$, $x \in$ **Dog Breeds**

Example: Existential Quantification

- Express this statement in logic:

Every last one of the large fluffy dog breeds are cute.

Let $L(x)$: x **is large**, $x \in$ **Dog Breeds**

Let $F(x)$: x **is fluffy**, $x \in$ **Dog Breeds**

Let $C(x)$: x **is cute**, $x \in$ **Dog Breeds**

$\forall x \, [(L(x) \wedge F(x)) \rightarrow C(x)]$, $x \in$ **Dog Breeds**

If a dog is both large and fluffy, then it is cute.

(vs. … $\wedge$ $C(x)$: **All** dog breeds are large, fluffy and cute

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Typically $\forall$ pairs with $\rightarrow$ and $\exists$ goes with $\wedge$

Playposit Question

Which of the below statements correctly translates the following into logic: “The temperature gets above 105 on some summer days”

Let $T(x)$: the temperature got above 105 on day x,

$x \in$ Day of the year

$S(x)$: day x is in the summer, $x \in$ Day of the year

A. $\exists x(T(x) \wedge S(x))$, $x \in$ Days of the year
B. $\forall x(T(x) \rightarrow S(x))$, $x \in$ Days of the year
C. $\forall x(T(x) \wedge S(x))$, $x \in$ Days of the year
D. $\exists x(S(x) \rightarrow T(x))$, $x \in$ Days of the year

Converting From English to Quantified Predicates

Example 1: Express the following statement using Logic

"Some people in this class have seen Star Wars"

1. What are our predicates and their domains?

$S(x)$: x has seen Star Wars, $x \in$ People

2. What is our domain?

People in this class

2b. Does our domain create new predicates?

Yes! $C(x)$: x is in this class, $x \in$ People

3. What quantifier do we use?

$\exists x$

Converting From English to Quantified Predicates

Example 1: Express the following statement using Logic

"Some people in this class have seen Star Wars"

Putting it all together:

$S(x)$: x has seen Star Wars, $x \in$ People

$C(x)$: x is in this class, $x \in$ People

$$\exists x\,(C(x) \wedge S(x)),\ \ x \in \textbf{People}$$

Converting From English to Quantified Predicates

Example 2: Express the following statement using Logic

"All people in this class who have seen Star Wars think it's great"

1. What are our predicates and their domains?

$S(x)$: x has seen Star Wars, $x \in$ People

2. What is our domain?

People in this class

2b. Does our domain create new predicates?

Yes! $C(x)$: x is in this class, $x \in$ People

3. What quantifier do we use?

$\forall x$

Converting From English to Quantified Predicates

Example 2: Express the following statement using Logic

"All people in this class who have seen Star Wars think it's great"

Putting it all together:

$S(x)$: x has seen Star Wars, $x \in$ People

$G(x)$: x thinks Star Wars is great, $x \in$ People

$C(x)$: x is in this class, $x \in$ People

$$\forall x \; ((C(x) \wedge S(x)) \rightarrow G(x)), \;\; x \in \textbf{People}$$

Converting From Quantified Predicates to English

Example 1: Express the following statement in English

"$\forall x\,(C(x) \rightarrow (P(x) \wedge J(x))),\ x \in$ People"

Where $J(x)$: x knows Java

$P(x)$: x knows Python

$C(x)$: x is in this class

Everyone in this class knows Python and Java

Converting From Quantified Predicates to English

Example 2: Express the following statement in English

"$\forall x\,(C(x) \wedge P(x) \wedge J(x)),\ x \in$ People"

Where $J(x)$: x knows Java

$P(x)$: x knows Python

$C(x)$: x is in this class

All people are in this class and know Python and Java

Converting From Quantified Predicates to English

Example 3: Express the following statement in English

"$\exists x\,(C(x) \rightarrow (P(x) \wedge J(x)))$, $x \in$ People"

Where $J(x)$: x knows Java

$P(x)$: x knows Python

$C(x)$: x is in this class

For some person, if they are in this class, then they know Python and Java

Converting From Quantified Predicates to English

Example 4: Express the following statement in English

"$\exists x\,(C(x) \wedge P(x) \wedge J(x)),\ x \in$ People"

Where $J(x)$: x knows Java

$P(x)$: x knows Python

$C(x)$: x is in this class

Someone in this class knows Python and Java

Nested Quantifiers

- Sometimes we may need to use multiple quantifiers
  - We can't express "Everybody loves someone" using a single quantifier.
  - Suppose predicate **loves**(x, y)**:** "Person x loves person y"

Nested Quantifiers

- **loves**(x, y)**:** "Person x loves person y"

- The four possible nestings:

Same quantifiers

- $\forall x \forall y$ **loves**(x, y)
- $\exists x \exists y$ **loves**(x, y)

Mixed quantifiers

- $\exists x \forall y$ **loves**(x, y)
- $\forall x \exists y$ **loves**(x, y)

Evaluating Nested Quantifiers of the Same Type

- Example: **loves**(x, y)**:** "Person x loves person y"
  - $\forall x \forall y$ **loves**(x, y)
    - "Everyone loves everyone".
  - $\exists x \exists y$ **loves**(x, y)
    - "There is someone who loves someone else" (or possibly themself!)

Evaluating Mixed Quantifiers

- Example: **loves**(x, y)**:** "Person x loves person y"
  - $\exists x \forall y$ **loves**(x, y)
    - "There is someone who loves everyone"
  - $\forall x \exists y$ **loves**(x, y)
    - "Everyone loves at least one person (possibly themself!)"

Evaluating Mixed Quantified

- Distinguishing $\exists x \forall y\ S(x, y)$ from $\forall i \exists k\ T(i, k)$
  - $\exists x \forall y\ S(x, y)$ - "There exists an x such that, for every y, $S(x, y)$ is true."
    - Somewhere in x's domain is an x that can be paired with any y's domain to make $S(x, y)$ true.
  - $\forall i \exists k\ T(i, k)$- "For any i there exists a k such that $T(i, k)$ is true.
    - No matter which i is selected, we can find some k to pair with the i to make $T(i, k)$ true. (Note that the k may vary with the i)

Evaluating Mixed Quantified

- Example:
  - Given $P(x, y) : x - y = 0, x, y \in \mathbb{Z}$
  - Evaluate: $\exists x \forall y \; P(x, y)$

Evaluating Mixed Quantified

- Example:
  - Given $P(x, y) : x - y = 0, x, y \in \mathbb{Z}$
  - Evaluate: $\exists x \forall y \; P(x, y)$ = **False**
    - (No such magical integer x exists!)

x	y	$P(x, y)$
0		
0	$\vdots$	$\vdots$
0		
0	-1	F
0	0	T
0	1	F
0		
0	$\vdots$	$\vdots$
0		

Evaluating Mixed Quantified

- Example:
  - Given $P(x, y) : x - y = 0, x, y \in \mathbb{Z}$
  - Evaluate: $\exists x \forall y \ P(x, y)$ = **False**
  - Evaluate: $\forall x \exists y \ P(x, y)$ = **True**
    - (No matter the x, there's an integer y ($y = x$) that makes $P(x, y)$ true.)

x	y	$P(x, y)$
-3	-3	T
-2	-2	T
-1	-1	T
0	0	T
1	1	T
2	2	T
3	3	T

Playposit Question

Which of the following statements are true?

- $\exists x \forall y (x < y),\ x \in \mathbb{Z}, y \in \mathbb{Z}^+$
- $\forall x \forall y\, x$ weighs less than $y, x \in$ animal, $y \in$ car
- $\forall y \exists x\ \ x$ weighs more than $y, x \in$ animal, $y \in$ car
- $\exists x \exists y\ x * y = 10,\ x \in \{1,2,3,4\}, y \in \mathbb{Z}$

Converting From English to Nested Quantifiers

Example 1: Express the following statement using Logic

If $x < y$ where $x, y \in \mathbb{R}$, then $ax < ay$, $a \in \mathbb{R}$

1. What are our predicates and their domains?

$P(x, y) : x < y,\ x, y \in \mathbb{R}.\ Q(a, x, y) : ax < ay,\ x, y, a \in \mathbb{R}$

2. What is our domain?

$\mathbb{R}$

2b. Does our domain create new predicates?

No.

3. What quantifier(s) do we use?

$\forall x, \forall y, \forall a$

Converting From English to Quantified Predicates

Example 1: Express the following statement using Logic

If $x < y$ where $x, y \in \mathbb{R}$, then $ax < ay$, $a \in \mathbb{R}$

Putting it all together:

$P(x, y) : x < y,\ x, y \in \mathbb{R}.$
$Q(a, x, y) : ax < ay,\ x, y, a \in \mathbb{R}$

$$\forall x \forall y \forall a\, (P(x, y) \rightarrow Q(a, x, y)),\ x, y, a \in \mathbb{R}$$

Note: The truth value of this statement is false. For this statement to be true, a needs to be positive!

Converting From English to Nested Quantifiers

Example 2: Express the following statement using Logic

"The difference of two positive integers

is not necessarily positive"

1. What are our predicates and their domains?

$P(x, y) : x - y > 0,\ x, y \in \mathbb{R}.\ Q(x) : x > 0,\ x \in \mathbb{R}$

2. What is our domain?

$\mathbb{Z}$

2b. Does our domain create new predicates?

No.

3. What quantifier(s) do we use?

$\exists x, \exists y$

Converting From English to Nested Quantifiers

Example 2: Express the following statement using Logic

"The difference of two positive integers
is not necessarily positive"

Putting it all together:

$P(x, y) : x - y > 0,\ x, y \in \mathbb{R}.$
$Q(x) : x > 0,\ x \in \mathbb{R}$

$$\exists x \exists y\, (Q(x) \wedge Q(y) \wedge \neg P(x, y)),\ x, y \in \mathbb{Z}$$

Converting From English to Nested Quantifiers

Example 2: Express the following statement using Logic

"The difference of two positive integers
is not necessarily positive"

A simpler version:

$P(x, y) : x - y > 0, \; x, y \in \mathbb{R}.$

Change our domain to $\mathbb{Z}^+$

$$\exists x \exists y \neg P(x, y), \; x, y \in \mathbb{Z}^+$$

Converting From Nested Quantifiers to English

Example 1: Express the following statement in English

"$\exists x \forall y ((C(x) \land C(y)) \rightarrow F(x, y)),\ x, y \in$ People"

Where $C(x)$: x is in this class, $x \in$ People

$F(x, y)$: x and y are friends, $x, y \in$ People

Someone in this class is friends with everyone else in this class

Converting From Nested Quantifiers to English

Example 2: Express the following statement in English

"$\forall x \forall y\, ((C(x) \wedge C(y)) \rightarrow F(x, y)),\ x, y \in$ People"

Where $C(x)$: x is in this class, $x \in$ People

$F(x, y)$: x and y are friends, $x, y \in$ People

Everyone in this class is friends with everyone in this class

Converting From Nested Quantifiers to English

Example 3: Express the following statement in English

"$\exists x \, \exists y \, (C(x) \wedge C(y) \wedge F(x, y)), \; x, y \in$ People"

Where $C(x)$: x is in this class, $x \in$ People

$F(x, y)$: x and y are friends, $x, y \in$ People

Two people in this class are friends.

Note: the two people don't have to be different, they could be the same person

Converting From Nested Quantifiers to English

Example 4: Express the following statement in English

"$\forall x\,(C(x) \rightarrow \exists y\,(C(y) \land F(x, y))),\; x, y \in$ People"

Where $C(x)$: x is in this class, $x \in$ People

$F(x, y)$: x and y are friends, $x, y \in$ People

Everyone in this class is friends with someone in this class

Negation of Quantified Expressions

- Remember De Morgan's Laws for Propositions? Well…

Definition: *Generalized De Morgan's Laws*

The Generalized De Morgan's Laws are the pair of equivalences:

$$\neg \forall x P(x) \equiv \exists \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall \neg P(x)$$

Demonstration: $\neg \forall x\, P(x) \equiv \exists x \neg P(x)$

- **Reminder:**

$$\forall x\, S(x), x \in D \equiv S(d_1) \wedge S(d_2) \wedge S(d_3)\ldots$$

$$\exists x\, S(x), x \in D \equiv S(d_1) \vee S(d_2) \vee S(d_3)\ldots$$

Demonstration: $\neg \forall x\, P(x) \equiv \exists x \neg P(x)$

- Let $S(x)$: x is cute, $x \in D$. Let $D =$ {all dog breeds}

$$\forall x\, S(x), x \in D \equiv S(d_1) \wedge S(d_2) \wedge S(d_3) \ldots$$

$$\begin{aligned} \neg \forall x\, S(x), x \in D &\equiv \neg(S(d_1) \wedge S(d_2) \wedge S(d_3) \ldots) \\ &\equiv \neg S(d_1) \vee \neg S(d_2) \vee \neg S(d_3) \ldots \\ &\equiv \exists x\ \neg S(x),\ x \in D \end{aligned}$$

$$\exists x\, S(x), x \in D \equiv S(d_1) \vee S(d_2) \vee S(d_3) \ldots$$

Negation of Nested Quantifiers

- Apply De Morgan's Laws from left to right
- Example:

$$\neg(\exists x \forall y (G(x) \vee \neg H(y)))$$

$$\equiv \forall x \neg(\forall y (G(x) \vee \neg H(y)))$$ **[General DeMorgan]**

$$\equiv \forall x \exists y \neg(G(x) \vee \neg H(y))$$ **[General DeMorgan]**

$$\equiv \forall x \exists y (\neg G(x) \wedge H(y))$$ **[DeMorgan]**

Converting From Quantified Predicates to Propositional Logic

Example 1:

Let $P(x)$: x is a prime number, $x \in \mathbb{Z}$

Express $\neg \forall x\, P(x),\ x \in \{3,5,7,9,11\}$ as quantified predicates and in propositional logic.

Quantified Predicate: $\exists x\ \neg P(x)$

Propositional Logic: $\neg(P(3) \wedge P(5) \wedge P(7) \wedge P(9) \wedge P(11))$

$$\equiv \neg P(3) \vee \neg P(5) \vee \neg P(7) \vee \neg P(9) \vee \neg P(11)$$

What is its truth value?

Converting From Quantified Predicates to Propositional Logic

Example 2:

Let $P(x)$: x is a prime number, $x \in \mathbb{Z}$

Express $\neg \exists x\, P(x)$, $x \in \{3,5,7,9,11\}$ as quantified predicates and in propositional logic.

Quantified Predicate: $\forall x\ \neg P(x)$

Propositional Logic: $\neg(P(3) \vee P(5) \vee P(7) \vee P(9) \vee P(11))$

$$\equiv \neg P(3) \wedge \neg P(5) \wedge \neg P(7) \wedge \neg P(9) \wedge \neg P(11)$$

Playposit Question

Which of the following properly negates the below statement:

"Blue is better than all other colors"

A. All other colors are better than blue
B. Green is better than blue
C. At least one color is better than blue
D. Blue is better than at least one color
E. Blue is only better than green

Expressing "Exactly one…" Statements

- Consider the conversational (& correct!) English statement

  Only one citizen of Montana is a member of the U.S. House of Representatives

- And consider this awkward but useful rewording:

  A member of the US House of Representatives exists in the set of citizens of Montana, and, if anyone in Montana is a member of the House, that person is the representative

Expressing "Exactly one…" Statements

- That rewording is useful because it can be directly expressed logically:

  $R(x)$: x is a member of the US House of Representatives, $x \in$ **People**

  $\exists x(R(x) \wedge \forall y[R(y) \rightarrow (y = x)]), x, y \in$ **Citizens of Montana**

This domain should be simplified, but using it makes the logic easier to read (for now)

Expressing "Exactly one…" Statements

- That rewording is useful because it can be directly expressed logically:

  $R(x)$: x is a member of the US House of Representatives, $x \in$ **People**

  $\exists x(R(x) \land \forall y[R(y) \rightarrow (y = x)]), x, y \in$ **Citizens of Montana**

- Interpretation: (At least one) $\land$ (No more than one)

- $\therefore$ Impossible for there to be two representatives!

Expression "Exactly two…" Statement

- Key observation:

  Exactly 2 ≡ At least 2 ∧ At most 2

  $$(n = 2) \equiv (n \geq 2) \land (n \leq 2)$$

- Awkward English:

  At least two citizens of Montana are U.S. Senators, and at most two citizens of Montana are U.S. Senators.

- Better:

  Exactly two citizens of Montana are U.S. Citizens

Expression "Exactly two…" Statement

- Consider the two halves separately:

  1. "At least two citizens of Montana are U.S. Senators"

  2. "At most two citizens of Montana are U.S. Senators"

Expressing “At least two”

- How do we express “At least two citizens of Montana are U.S. Senators”? Let $S(x)$: x is a U.S. Senator, $x \in$ People

- Why doesn’t this work?

  - $\exists x \exists y \, (S(x) \wedge S(y))$, $x, y \in$ Citizens of Montana **(x, y could be identical)**

- Correct version:

  - $\exists x \exists y \, (S(x) \wedge S(y) \wedge (x \neq y))$, $x, y \in$ Citizens of Montana

x and y are both U.S. Senators

and

they are not the same person

Expression "Exactly two…" Statement

- Consider the two halves separately:

1. "At least two citizens of Montana are U.S. Senators"

   $S(x)$: x is a U.S. Senator, $x \in$ **People**

   $\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y)),$

   $x, y \in$ **Citizens of Montana**

2. "At most two citizens of Montana are U.S. Senators"

Expressing “At most two”

- How do we express “At most two citizens of Montana are U.S. Senators”? Let $S(x)$: x is a U.S. Senator, $x \in$ People

- Start with “at most one”:
  - $\forall x \forall y \, (S(x) \land S(y)) \rightarrow (x = y)$

- Extended to 2 (i.e. “at most two”):
  - $$\forall x \forall y \forall z \, ((S(x) \land S(y) \land S(z)) \rightarrow ((x = y) \lor (x = z) \lor (y = z)),$$
    $x, y \in$ Citizens of Montana

If x, y, z are all U.S. Senators…

… then there is **at least** one pair which are the same person

Expression "Exactly two…" Statement

- Consider the two halves separately:

1. "At least two citizens of Montana are U.S. Senators"

   $S(x)$: x is a U.S. Senator, $x \in$ **People**

   $\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y)),$

   $x, y \in$ **Citizens of Montana**

2. "At most two citizens of Montana are U.S. Senators"

   $\forall x \forall y \forall z ((S(x) \wedge S(y) \wedge S(z)) \rightarrow$

   $(x = y \vee y = z \vee x = z))$

   $x, y, z \in$ **Citizens of Montana**

Expression “Exactly two…” Statement

- Finally, **AND** together

$$\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y))$$

- and

$$\forall x \forall y \forall z ((S(x) \wedge S(y) \wedge S(z)) \rightarrow (x = y \vee y = z \vee x = z))$$

Expression "Exactly two..." Statement

- Finally, **AND** together

$$\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y))$$

- and

$$\forall x \forall y \forall z ((S(x) \wedge S(y) \wedge S(z)) \rightarrow$$
$$(x = y \vee y = z \vee x = z))$$

$$\exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y) \wedge$$
$$\forall z [S(z) \rightarrow (z = x \vee z = y)]),$$
$x, y, z \in$ **Citizens of Montana**

Why is the second half simplified?

Playposit Question

Which of the following quantifications corresponds to “At least three”. Let $S(x)$: x is on the coast, $x \in$ States

A. $\exists x \exists y \exists z \ (S(x) \wedge S(y) \wedge S(z) \wedge (x \neq y) \wedge (x \neq z))$,
$x, y, z \in$ States

B. $\exists x \exists y \exists z \ (S(x) \wedge S(y) \wedge S(z) \wedge (x \neq y) \wedge (x \neq z) \wedge (y \neq z))$
$x, y, z \in$ States

C. $\forall x \forall y \forall z \forall a \ ((S(x) \wedge S(y) \wedge S(z) \wedge S(a)) \rightarrow (a = x \vee a = y \vee a = z))$
$x, y, z, a \in$ States

D. $\forall x \forall y \forall z \forall a \ ((S(x) \wedge S(y) \wedge S(z) \wedge S(a)) \rightarrow (a = x \vee a = y \vee a = z$
$\vee\, x = y \vee x = z \vee y = z))$, $x, y, z, a \in$ States

Reminders

- Homework 2 due **this** Friday (06/19)
- Grades for Homework 1 should be posted before Friday
- Quiz 1 **this** Tuesday (on Logic)