Relations

Section 9.1, 9.3, 9.5, 9.6

Background

- Having collections of data: Good
- Knowing the connections between collections: Better!
- **Example**:
  - Students - Courses
  - Businesses - Email Adresses
  - Dogs - Trees

Relations

Definition: *(Binary) Relation*

A binary relation from set X to Y is a subset of the Cartesian Product of X (the domain) and Y (the codomain).

NOTE: a relation "on set W" $\equiv$ "from set W to set W".

Example:

$$A = \{\textbf{Leslie Knope, Jim Halpert, Michael Scott, Ann Perkins, Ben Wyatt}\}$$

$$B = \{\textbf{Parks and Rec, The Office}\}$$

$$R = \{(\textbf{Leslie Knope, Parks and Rec}), (\textbf{JH, O}), (\textbf{MS, O}), (\textbf{AP, P\&R}), (\textbf{BW, P\&R})\}$$

Relations

Definition: *Related*

If $(x, y) \in R$, x is **related to** y ($x \, R \, y$)

Example:

$($**Leslie Knope, Parks and Rec**$) \in R$ $($**Leslie Knope** R **Parks and Rec**$)$

$($**Jim Halpert, Parks and Rec**$) \notin R$ $($**Jim Halpert** $\not R$ **Parks and Rec**$)$

Let $H = \{1,2,3,4,5,6\}$ and let R be a relation on H such that $x \ R \ y$ when $x \,\%\, y = 0$, $x \neq y$

$R = \{(2,1), (3,1), (4,1), (5,1), (6,1), (4,2), (6,2), (6,3)\}$

Defining Relations

- Relations are just sets of ordered pairs
  - We can use set builder notation to define sets

Example:

$A = \{$**Leslie Knope, Jim Halpert, Michael Scott, Ann Perkins, Ben Wyatt**$\}$

$B = \{$**Parks and Rec, The Office**$\}$

$R = \{(a, b) \,|\, a \in A \wedge b \in B \wedge a$ **is a character in** $b\}$

Let $H = \{1,2,3,4,5,6\}$ and we want a relation on H such that $x \; R \; y$ when $x \,\%\, y = 0, \, x \neq y$

$R = \{(x, y) \,|\, x, y \in H \wedge x \,\%\, y = 0\}$

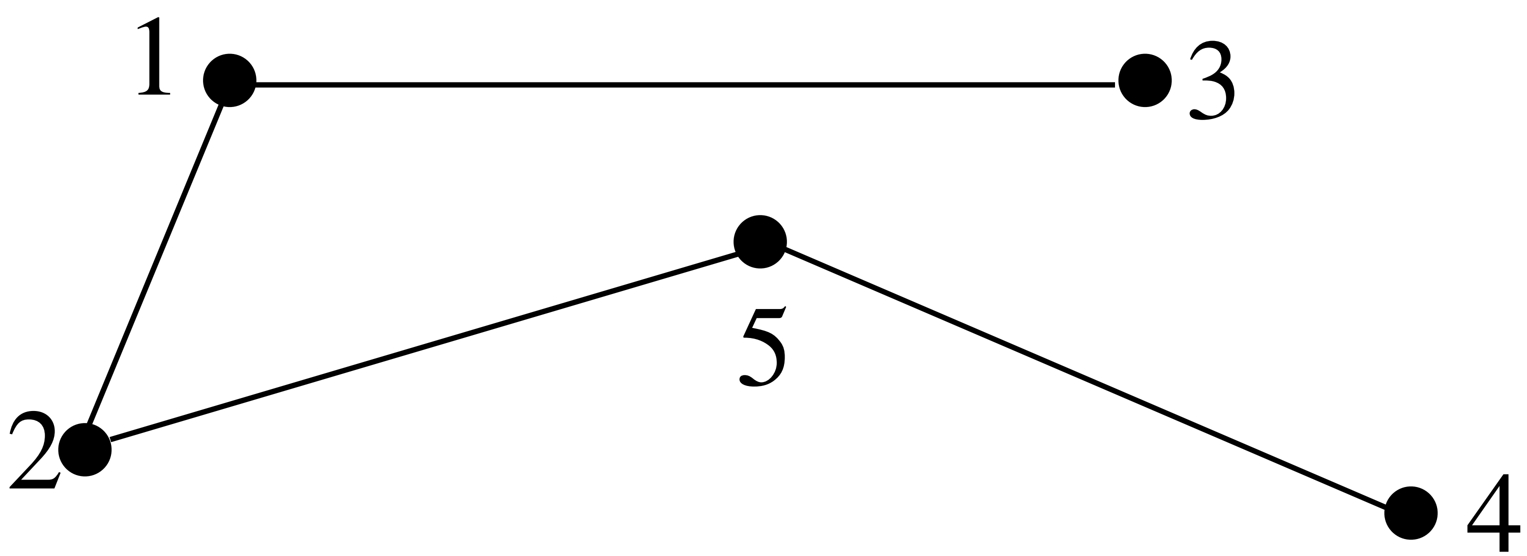
Playposit

- Let $A = \{2,4,6\}$. Let R be the relation on A where $(x, y) \in R$ when $\dfrac{x}{y} \in \mathbb{Z}$. Which of the following ordered pairs are in R?

• (2,2)	• (2,4)	• (2,6)
• (4,2)	• (4,4)	• (4,6)
• (6,2)	• (6,4)	• (6,6)

Graph Representations of Relations

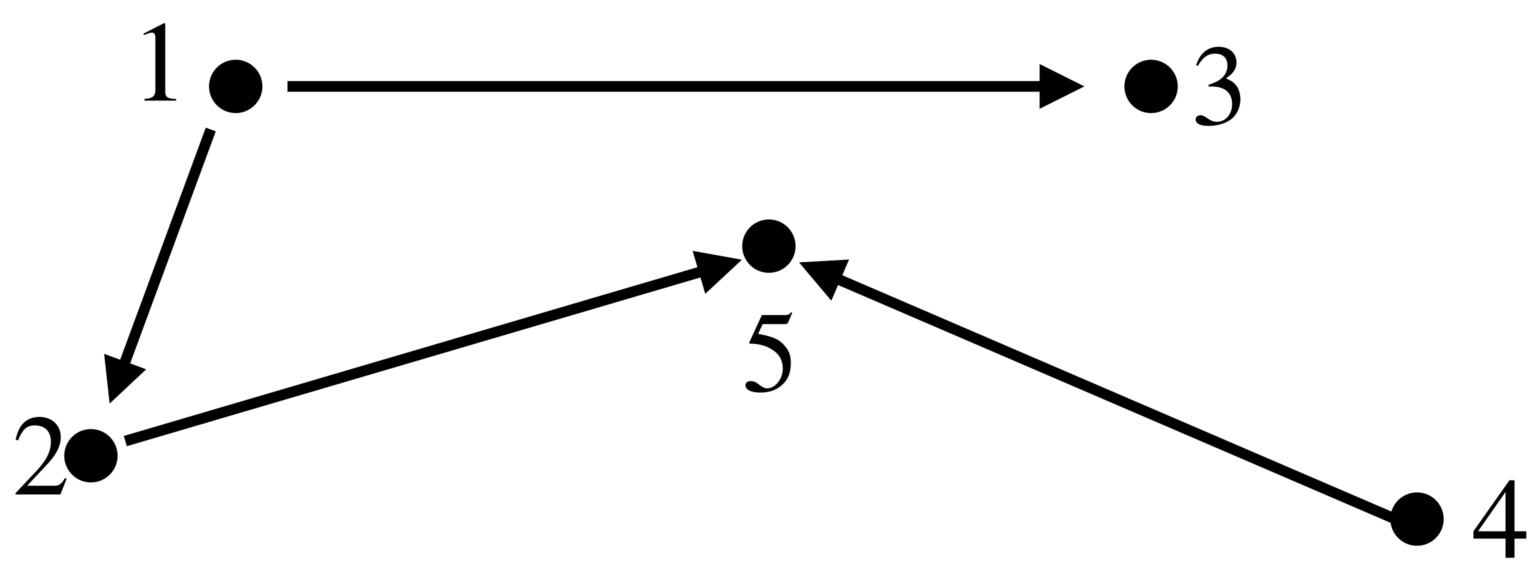
- **Graphs**:
  - A set V of vertices (nodes) and a set E of pairs of vertices that represent an edge between those two vertices



$$V = \{1,2,3,4,5\}$$
$$E = \{(1,3), (1,2), (2,5), (4,5)\}$$

Graph Representations of Relations

- **Directed Graphs (Digraph)**:
  - A set V of vertices (nodes) and a set E of pairs of vertices that represent an edge between those two vertices
  - In edge (a,b), a is the initial vertex and b is the terminal vertex



$$V = \{1,2,3,4,5\}$$
$$E = \{(1,3), (1,2), (2,5), (4,5)\}$$

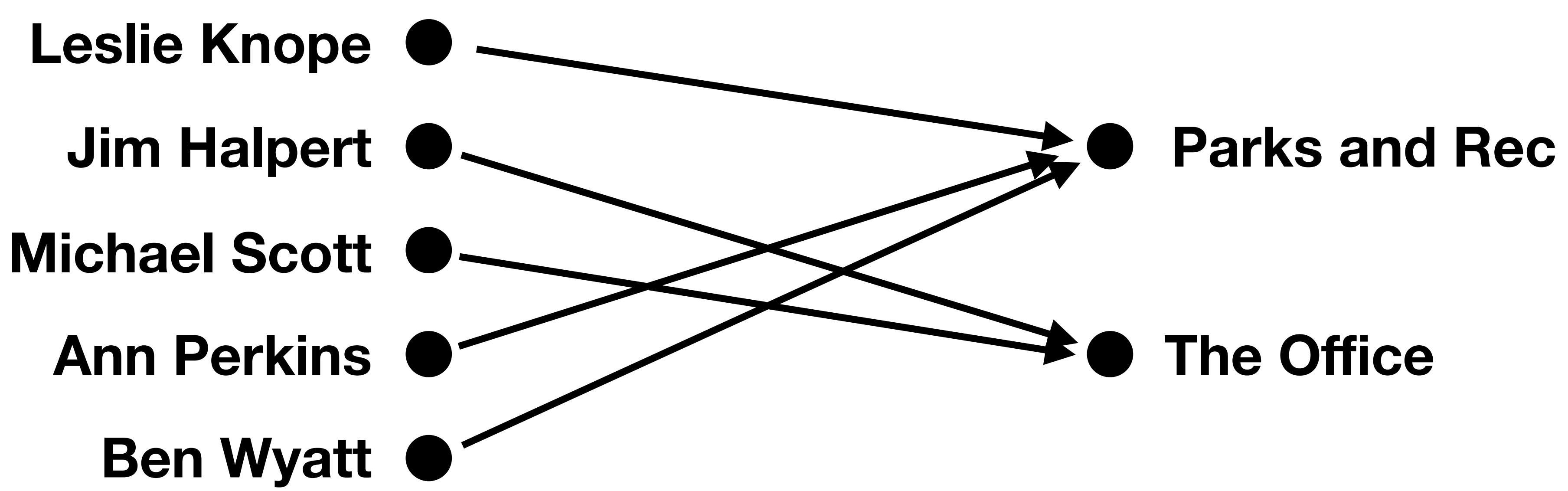
Graph Representations of Relations

- **Example:**

$$A = \{\textbf{Leslie Knope, Jim Halpert, Michael Scott,}$$
$$\textbf{Ann Perkins, Ben Wyatt}\}$$

$$B = \{\textbf{Parks and Rec, The Office}\}$$

$$R = \{(\textbf{Leslie Knope, Parks and Rec}), (\textbf{JH, O}),$$
$$(\textbf{MS, O}), (\textbf{AP, P\&R}), (\textbf{BW, P\&R})\}$$

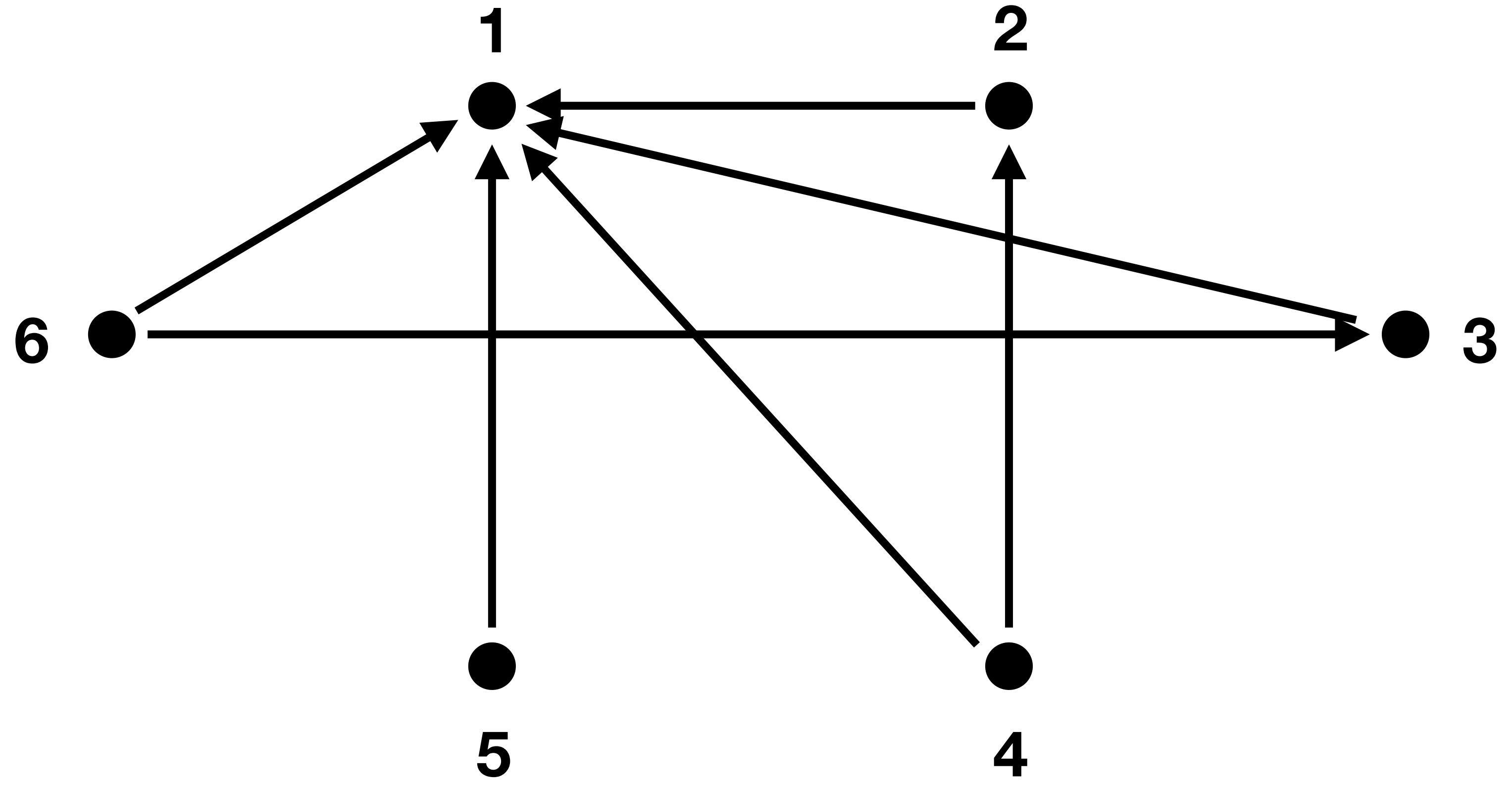


Graph Representations of Relations

- **Example:** $x \% y = 0,\ x \neq y$

Recall: $H = \{1,2,3,4,5,6\}$

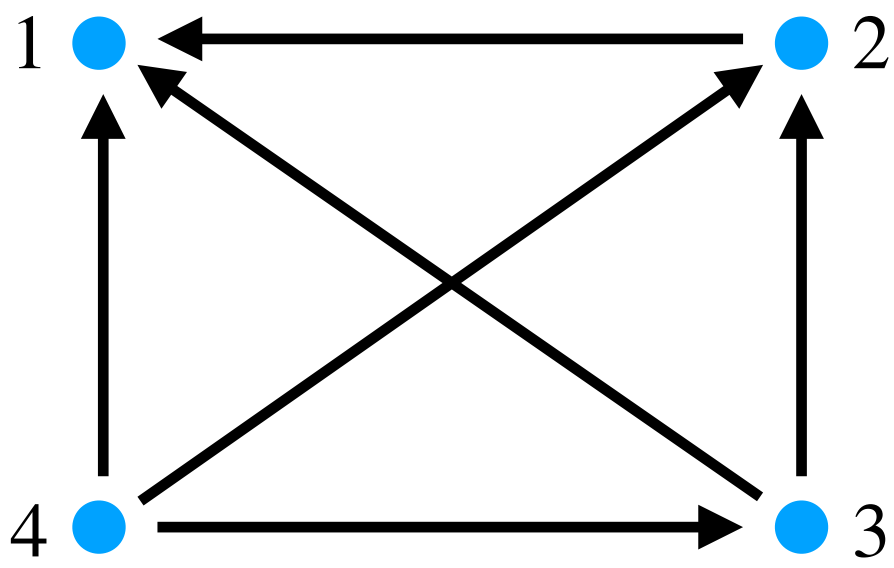
$R = \{(2,1), (3,1), (4,1), (5,1), (6,1), (4,2), (6,2), (6,3)\}$



Note: Vertices with just one outgoing edge are prime

Playposit

Let $A = \{1,2,3,4\}$. What relation does the below graph represent?



A. $R = \{(x, y) \,|\, x - y > 0 \wedge x, y \in H\}$
B. $R = \{(x, y) \,|\, xy < 5 \wedge x, y \in H\}$
C. $R = \{(x, y) \,|\, x + y \geq 6 \wedge x, y \in H\}$
D. $R = \{(x, y) \,|\, (x + y) \,\%\, 2 = 0 \wedge x, y \in H\}$

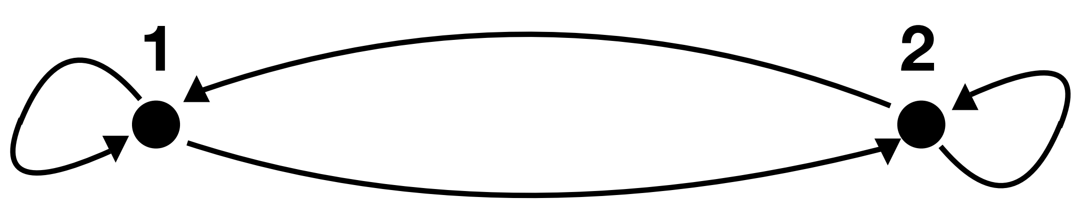
Properties of Relations

Definition: *Reflexivity*

A relation R on set A is reflexive when $(a, a) \in R$, $\forall a \in A$

Example:

$$\{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$$



(A directed edge whose source is also the destination is a self-loop)

Properties of Relations

Definition: *Symmetry*

A relation R on set A is symmetric if $(a, b) \in R$ whenever $(b, a) \in R$, for $a, b \in A$

(All non-self-loop edges are 'back-and-forth')

Example:

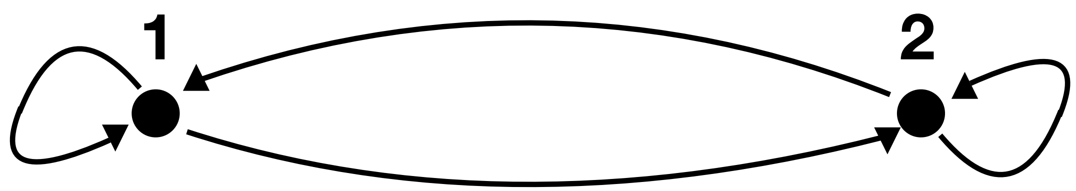
$R = \{(1,1), (1,2), (2,1), (2,2)\}$ **on** $A = \{1,2\}$ **is symmetric**

$R = \{(a, c), (a, d), (c, a), (b, c)\}$ **on** $A = \{a, b, c, d\}$ **is not symmetric (**(d, a) **and** (c, b) **are missing.)**

Properties of Relations

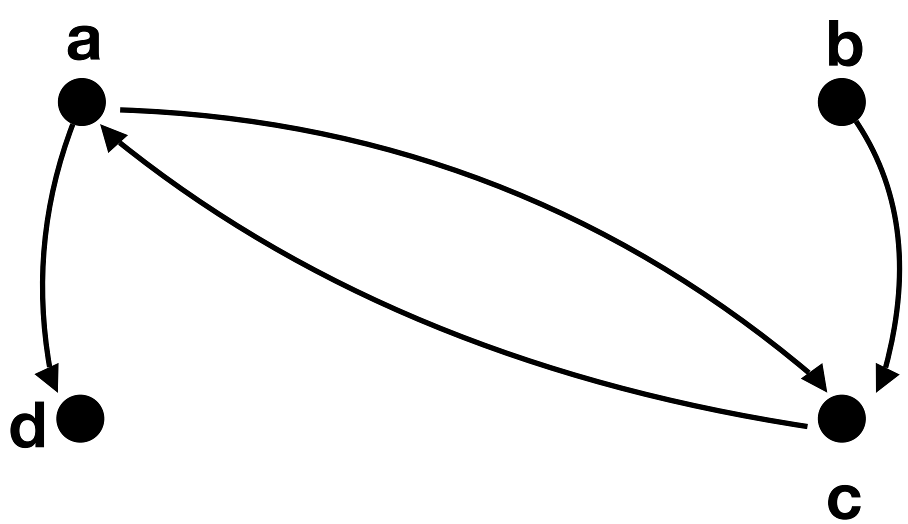
Example: Graph Representation & Symmetry

$R = \{(1,1), (1,2), (2,1), (2,2)\}$:



(Excepting self-loops, just have back-and-forth arrows)

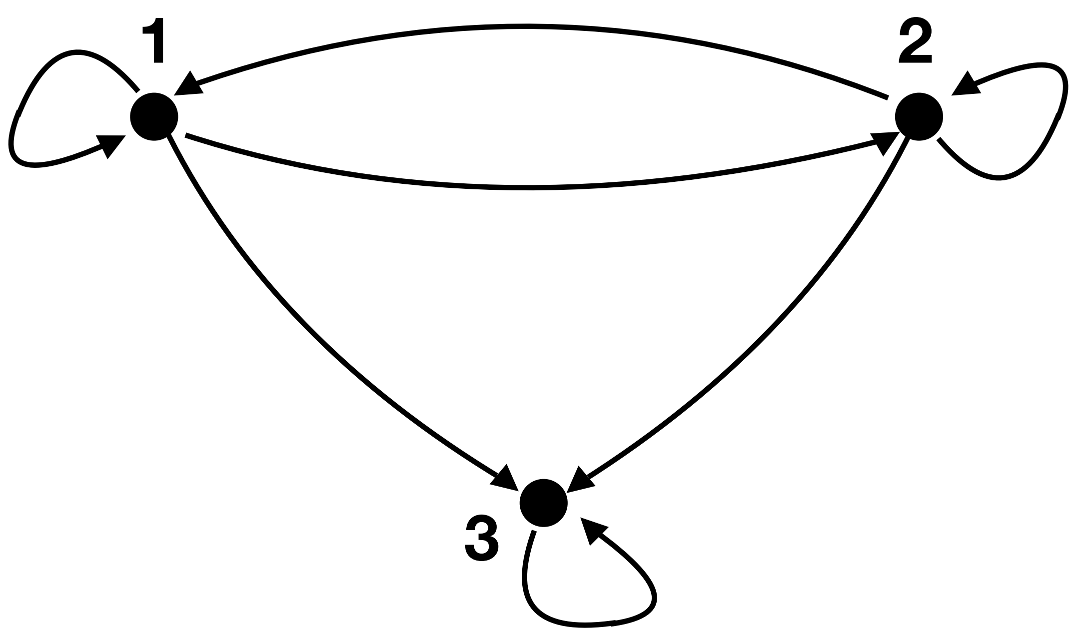
$R = \{(a, c), (a, d), (c, a), (b, c)\}$:



(Easy to see that (d, a) and (c, b) are missing

Playposit

Which of the following is true for the relation represented by the below graph?



A. It is only symmetric.
B. It is only reflexive.
C. It is both reflexive and symmetric.
D. It is neither reflexive nor symmetric.

Properties of Relations

Definition: *Antisymmetry*

A relation R on set A is antisymmetric if $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$, $\forall x, y \in A$.

(No non-self-loop edges are 'back-and-forth')

Example:

$\{(3,4)\}$ **on** $\{3,4\}$ **is antisymmetric (**$(4,3)$ **is not present)**

$\{(1,1), (3,1), (1,3)\}$ **on** $\{1,3\}$ **is not antisymmetric**

$\{(a, c), (a, d), (c, a), (b, c)\}$ **is not**

(Thus, relations may be neither symmetric nor antisymmetric.)

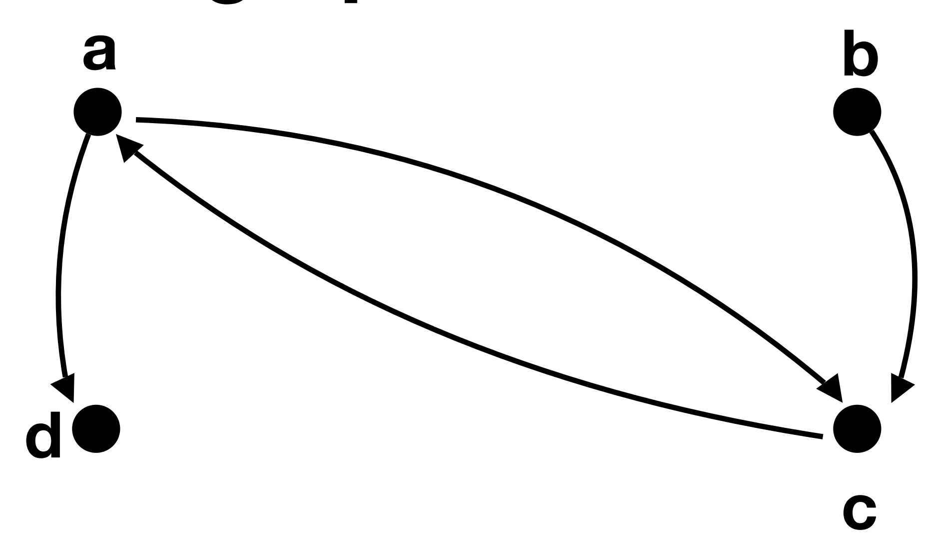
Properties of Relations

Example: Graph Representation & Antisymmetry

$R = \{(a,c),(a,d),(c,a),(b,c)\}$ **as a digraph**

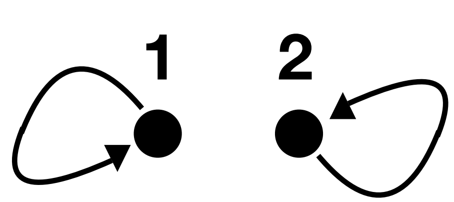
(The offending 'double edge'

Is easy to see)



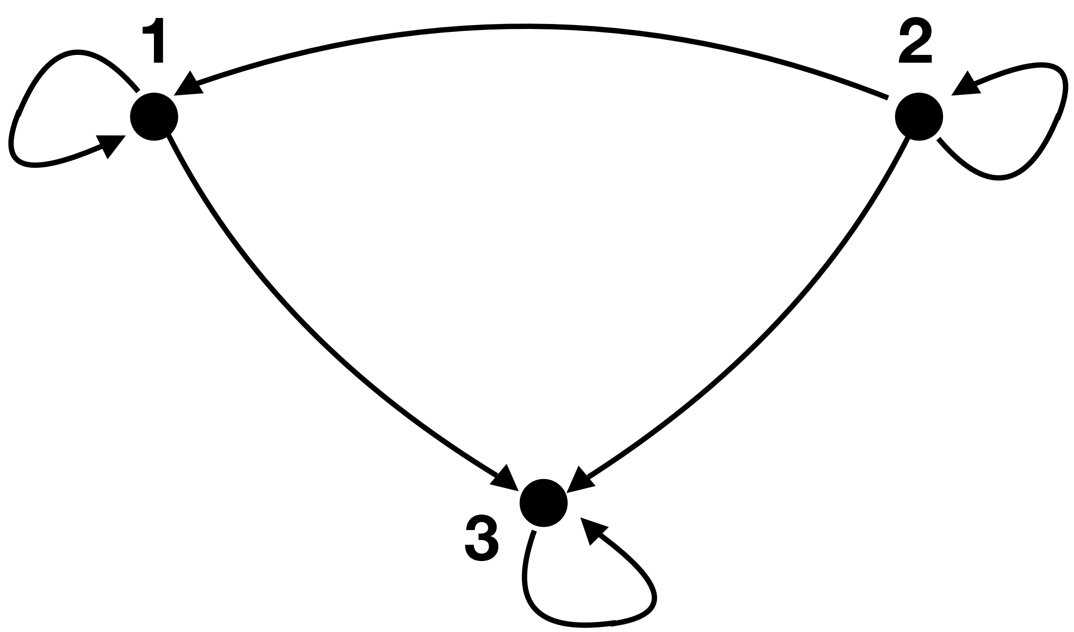
$R = \{(1,1),(2,2)\}$ **on** $\{1,2\}$

(both symmetric & antisymmetric) as a digraph:



Playposit

Which of the following is true for the relation represented by the below graph?



A. It is only symmetric.
B. It is only antisymmetric.
C. It is both symmetric and antisymmetric.
D. It is neither symmetric nor antisymmetric.

Properties of Relations

Definition: *Transitivity*

A relation R on set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, where $a, b, c \in A$

Example:

$R = \{(x, y), (x, z), (y, z), (z, x), (z, y)\}$ **on** $\{x, y, z\}$

(x, y) & $(y, z) \Rightarrow (x, z)$, **which is** $\in R$

(y, z) & $(z, y) \Rightarrow (y, y)$ **which is not in** R

$\therefore R$ **is not transitive**

Properties of Relations

Example:

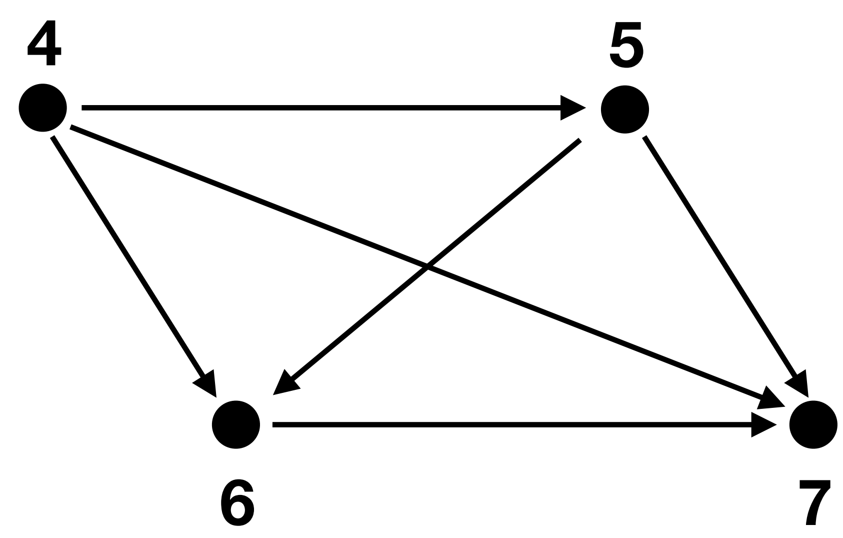
$S = \{(4,5), (4,6), (4,7), (5,6), (5,7), (6,7)\}$ **on** $\{4,5,6,7\}$

$(4,5) \;\&\; (5,6) \Rightarrow (4,6)$

$(4,5) \;\&\; (5,7) \Rightarrow (4,7)$

$(4,6) \;\&\; (6,7) \Rightarrow (4,7)$

$(5,6) \;\&\; (6,7) \Rightarrow (5,7)$



$\therefore S$ **is transitive**

(Note: Digraphs don't really help see transitivity)

Properties of Relations

Example:

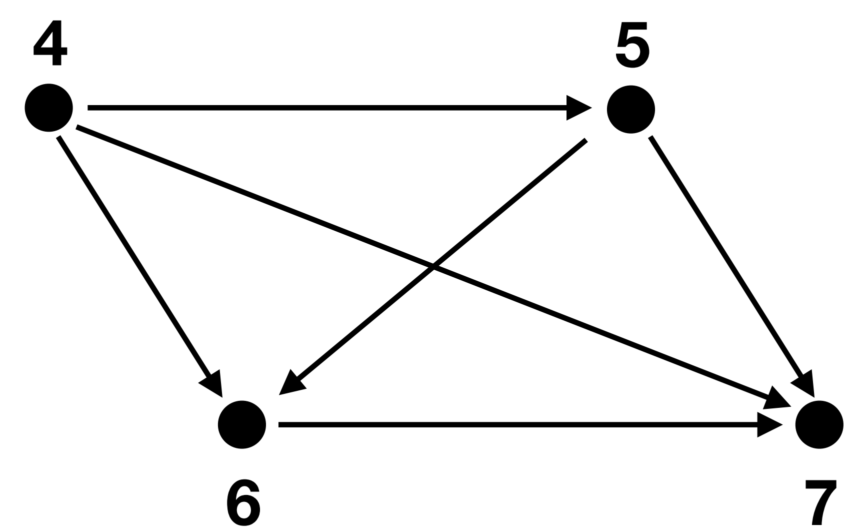
$S = \{(4,5), (4,6), (4,7), (5,6), (5,7), (6,7)\}$ **on** $\{4,5,6,7\}$

$(4,5) \,\&\, (5,6) \Rightarrow (4,6)$

$(4,5) \,\&\, (5,7) \Rightarrow (4,7)$

$(4,6) \,\&\, (6,7) \Rightarrow (4,7)$

$(5,6) \,\&\, (6,7) \Rightarrow (5,7)$



$\therefore S$ **is transitive**

(Note: Digraphs don't really help see transitivity)

Relational Composition Examples

- Three examples of creating relations from relations

- **Example #1:** Set operators

Recall: A relation is a set of ordered pairs

Let $A = \{1,2,3\}$

$R = \{(1,2), (1,3)\}$ **on** A

$S = \{(1,1), (2,3)\}$ **on** A

$R \cup S = \{(1,2), (1,3), (1,1), (2,3)\}$ **on** A

is also a relation on A

Relational Composition Examples

- **Example #2:** Swapping content of ordered pairs $(1,2) \Rightarrow (2,1)$

Definition: *Inverse*

The inverse of a relation R, denoted R^{-1}, contains all of the ordered pairs of R with their components exchanged

That is: $R^{-1} = \{(b,a) \,|\, (a,b) \in R\}$

Relational Composition Examples

- **Example #3:** Composites

Remember: $f \circ g = f(g(x))$

Definition: *Composite*

Let G be a relation from A to B, and F be a relation from B to C. The composite of F and G, $F \circ G$, is the relation of ordered pairs (a, c), $a \in A$, $c \in C$, such that $(a, b) \in G$ and $(b, c) \in F$, where $b \in B$

Example:

Let $X = \{(1,a), (2,b), (3,c)\}$

$Y = \{(1,2), (2,3), (1,3), (2,4)\}$

$X \circ Y = \{(1,b), (2,c), (1,c)\}$

Relational Composition Examples

- **Example #3:** Composites (cont)

Example:

Let $C = \{(\alpha, -4), (\alpha, -2), (\beta, -6), (\gamma, -4)\}$

$D = \{(q, \beta), (x, \alpha), (x, \gamma)\}$

$C \circ D = \{(q, -6), (x, -4), (x, -2)\}$

(Note that $(x, -4)$ is not repeated; this is a set)

Definition: *Complement*

The complement of a relation R, denoted $\overline{R}$, is $\{(a, b) \mid (a, b) \notin R\}$

Matrix Representation of Relations

- We assume that relations are on just one set
- The 0-1 matrix representation of relation R on set A is $|A| \times |A|$, with both dimensions labeled identically. When $(a, b) \in R$, then `matrix[a][b]=1`. Else, `matrix[a][b]=0`

Example:

$R = \{(a,c),(a,d),(c,a),(b,c)\}$ on $\{a,b,c,d\}$

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Matrix Representation of Relations

- **Observation #1:** Detecting Reflexivity
  - A relation is reflexive when its corresponding matrix representation has no 0's along the main diagonal

Example:

$R = \{(1,1), (1,2), (2,2)\}$ on $\{1,2\}$

$$\begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{1} \end{bmatrix}$$

The main diagonal is all 1's; $\therefore R$ **is reflexive**

Matrix Representation of Relations

- **Observation #2:** Detecting Symmetry
  - Let matrix M represent R. R is symmetric when $m_{ij} = 1$ iff $m_{ji} = 1$ is true

Example:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

When the relation is symmetric, the matrix is symmetric

Matrix Representation of Relations

- **Observation #3:** Detecting Transitivity
  - Let matrix M represent R. R is transitive when the non-zero elements of M^2 (or of $M^{([2])}$) are also non-zero in M

Example:

Is $\{(1,1),(2,2),(2,3),(3,2)\}$ on $\{1,2,3\}$ transitive?

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 2 & 1\\0 & 1 & 1\end{bmatrix}$$

So, no, the relation is not transitive

Matrix Representation of Relations

- **Observation #3:** Detecting Transitivity
  - Let matrix M represent R. R is transitive when the non-zero elements of M^2 (or of $M^{([2])}$) are also non-zero in M

Why does this work?

Consider $R = \{(1,1), (2,2), (2,3), (3,2)\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \boxed{0 \quad 1 \quad 0} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & \boxed{1} \end{bmatrix}$$

Everything 3 relates to

Everything related to 3

Can we transitively relate 3 to 3?

Equivalence Relations

Definition: *Equivalence Relation*

A relation on set A is an equivalence relation if it is *reflexive, symmetric,* and *transitive.*

Example:

$A = \{2,3,4\}$. Let Q be a relation on A such that $a = b$, $\forall a, b \in A$

Thus $Q = \{(2,2), (3,3), (4,4)\}$

Reflexive? Symmetric? Transitive?

Equivalence Relations

Example:

$B = \{-2, -1, 0, 1, 2\}$. Let R be a relation on B such that $|a| = |b|,\ \forall a, b \in B$

Thus $R = \{(0,0), (1,1), (1,-1), (-1,1), (-1,-1),$
$(2,2), (2,-2), (-2,2), (-2,-2)\}$

This is also an equivalence relation.

Playposit

Is the following relation on $\{a, b, c\}$ an equivalence relation (if no, select the answer that explains why)?

$$R = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, b), (c, c)\}$$

A. Yes, it is an equivalence relation.
B. No, because it is not transitive.
C. No, because it is not symmetric.
D. No, because it is not reflexive
E. None of the above

Equivalence Relations

So ... why are these called *equivalence* relations?

Recall:

$$R = \{(0,0),(1,1),(1,-1),(-1,1),(-1,-1),$$
$$(2,2),(2,-2),(-2,2),(-2,-2)\}$$

Note the "clusters" of 0's, 1's, and 2's. This gives a partition of the base set B :

$$\{\{0\},\{-1,1\},\{-2,2\}\}$$

Equivalence Relations

Definition: *Equivalence Class*

The equivalence relation R on set B, and an element $b \in B$, is $\{c \mid c \in B \wedge (b, c) \in R\}$ and is denoted $[b]$. (That is, the set of everything paired with & on the right side of b in R

Example: **(From previous slide)**

$[1] \Rightarrow \{c \mid c \in B \wedge (1,c) \in R\}$

R contains $(1,1)$ and $(1, -1)$, so $[1] = \{1, -1\}$.

Playposit

Which of the following is the equivalence class for [2]?

$$R = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$$

A. $\{1,2\}$
B. $\{1,3\}$
C. $\{2,3\}$
D. $\{1,2,3\}$

Partial Orders

- Consider scheduling the construction of a house.
- Example: foundation, then walls, then paint
  - But what about bathroom tile and the kitchen sink?

Definition: *Reflexive (a.k.a. Weak) Partial Order*

A relation R on set A is a (reflexive/weak) partial order if it is *reflexive, antisymmetric,* and *transitive.*

Notation: $x \preceq y$ **means** $(x, y) \in R$
when R **is a partial order**

Partial Orders

Example:

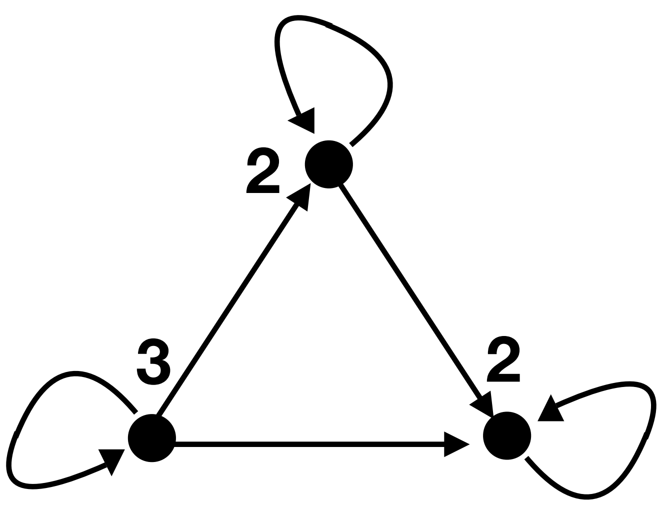
$S = \{(1,1), (1,2), (2,2), (3,1), (3,2), (3,3)\}$ **on** $\{1,2,3\}$

Is S reflexive?

Antisymmetric?

Transitive?

$\therefore$ S is a week partial order



$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

(Note: A partially-ordered set is called a *poset*)

Partial Orders

Definition: *Irreflexivity (of Relations)*

A relation R on set A is irreflexive if $\forall a \in A, (a, a) \notin R$
(Note: Not the same as “not reflexive”)

Definition: *Irreflexive (a.k.a. Strict) Partial Order*

A relation R on set A is a irreflexive partial order if it is *irreflexive, antisymmetric,* and *transitive.*

(Thus no self-loops allowed.)

Playposit

Which type of partial order (if any) is the following relation on $\{a, b, c\}$?

$$R = \{(a, a), (b, a), (c, b), (c, c)\}$$

A. Reflexive (a.k.a Weak) Partial order
B. Irreflexive (a.k.a Strict) Partial order
C. It is not a partial order.

Total Orders

Definition: *Comparable*

Let R be a weak partial order on set A. a and b are comparable if $a, b \in A$and either $a \preceq b$ or $b \preceq a$. (That is, $(a, b) \in R$ or $(b, a) \in R$)

Definition: *Total Order*

A weak-partial-ordered relation R on a set A is a total order if every pair of elements $a, b \in A$ are comparable. (Or: A relation R on A is a total order if R is antisymmetric, transitive, and comparable.)

Total Orders

Example:

$S = \{(1,1), (1,2), (2,2), (3,1), (3,2), (3,3)\}$ on $\{1,2,3\}$

It is a partial order and the pairs $(1,2)$, $(3,1)$ and $(3,2)$ show that all elements are comparable. $\therefore$ Total Order!

Let $T = \{(1,1), (1,3), (2,2), (2,3), (3,3)\}$ on $\{1,2,3\}$

Reflexive? Antisym? Transitive? $\therefore$ Partial Order

But: 1 and 2 are not comparable; this is <u>not</u> a total order.

Playposit

Which property does this relation on $\{a, b, c, d\}$ fail to satisfy in order to be a total order. ?

$$R = \{(a, a), (a, d), (b, b), (c, a), (c, b), (c, c), (c, d), (d, d)\}$$

A. Reflexive
B. Antisymmetric
C. Transitive
D. Comparable

Summary

- Equivalence Relation and Partial Orders Definitions
  - Equivalence Relation: **Reflexive** **Symmetric** **Transitive**
  - Weak Partial Order: **Reflexive** **Antisymmetric** **Transitive**
  - Strict Partial Order: **Irreflexive** **Antisymmetric** **Transitive**
  - Total Order: **Weak Partial Order + Comparability**

    or: comparable antisymmetric transitive