Sequences and Strings 4.3

Sequences

Definition: <u>Sequence</u> [1st Attempt]

An ordered list of items

Notation:

- Labels are lower-case letters
- Elements are subscripted: e_1, e_2, \ldots
- $\{e_n\} \Rightarrow e$ is an *n*-element sequence.

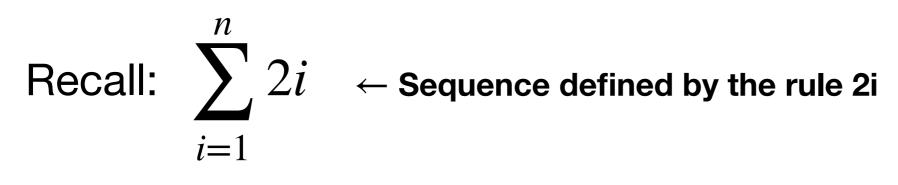
Example: Soup!

Cost sequence: s = 2,4,6,8,10,... (\$2 per can)

Soup Saturday: Buy 3 cans of soup, get on free!

s' = 2,4,6,6,8,10,12,12... (Not a set!)

Rules



Example:

 $s_n = 2n$ defines the original soup price sequence $n^2 + 1, n \ge 0$ defines the infinite sequence 1, 2, 5, 10, 17, ...

More notation:

Infinite sequences:

- 1. Ellipses (as in 1,2,5,10,17,...)
- 2. $\{d_n\}_{n=1}^{\infty}$

Sequences and Functions

Definition: <u>Sequence</u> [Final Version]

A sequence is the ordered range of a function from a set of integers to some set S**Example:**

o(n) = 2n - 1 on the domain $\{1,2,3,4,5\}$ defines the sequence 1,3,5,7,9

As a relation: $\{(1,1), (2,3), (3,5), (4,7), (5,9)\}$

Range of $\{1,3,5,7,9\}$

(Thus, the "ordered range" wording)

Arithmetic and Geometric Sequences

Definition: Arithmetic Sequence (a.k.a. Arithmetic Progression)

In an arithmetic sequence, the *common* difference $d = a_{n+1} - a_n$ is constant

Definition: <u>Geometric Sequence (a.k.a. Geometric Progression)</u> In a geometric sequence, the common ratio $r = \frac{g_{n+1}}{g_n}$ is constant

Example:

ln o: 1, 3, 5, 7,9 d = 2ln g: 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$..., $r = \frac{2}{3}$

Arithmetic Series

- The sum of the terms of an arithmetic sequence (a.k.a arithmetic series): $s_n = a_1 + \ldots + a_n = \frac{1}{2}n(a_1 + a_n)$
- Here's why: First, note that $a_n = a_1 + (n 1)d$.
- Next, here are two expressions for s_n :
 - $s_n = a_1 + (a_1 + d) + (a_1 + 2d) + \dots + (a_1 + (n 1)d)$

•
$$s_n = (a_n - (n-1)d) + (a_n - (n-2)d) + \dots + (a_n - d) + a_n$$

• Sum these expressions, and the d terms cancel, leaving:

•
$$2s_n = na_1 + na_n$$
 or $s_n = \frac{1}{2}n(a_1 + a_n)$

• Ex: In 1,3,5 $d = 2, a_1 = 1, a_1 + d = 3$ and $a_1 + 2d = 5$

Increasing Sequences

Definition: <u>Increasing Sequence</u>

An increasing sequence labeled i is ordered such that $i_n \leq i_{n+1}$.

Definition: <u>Non-Decreasing Sequence</u>

A non-decreasing sequence labeled i is ordered such that $i_n \leq i_{n+1}$ [Same as increasing!]

Definition: <u>Strictly Increasing Sequence</u>

A strictly increasing sequence labeled i is ordered such that $i_n < i_{n+1}$

Decreasing Sequences

Definition: <u>Decreasing Sequence</u>

A decreasing sequence labeled *i* is ordered such that $i_n \ge i_{n+1}$.

Definition: <u>Non-Increasing Sequence</u>

A non-increasing sequence labeled i is ordered such that $i_n \ge i_{n+1}$ [Same as decreasing!]

Definition: Strictly Decreasing Sequence

A strictly decreasing sequence labeled *i* is ordered such that $i_n > i_{n+1}$

Examples: Increasing/Decreasing Sequences

- The sequence g = 1,2,2,2,6,8,8,9 is:
 - Increasing
 - Non-Decreasing

•
$$h_n = \frac{1}{n}, \ 4 \le n \le 7$$
 $(h = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7})$

- $\{h_n\}_{n=4}^7$ is:
 - Decreasing, Non-Increasing
 - Strictly Decreasing

Subsequences

Definition: <u>Subsequence</u>

Sequence *x* is a subsequence of sequence *y* when the elements of *x* are found within *y* in the same relative order

Example:

Is
$$\frac{1}{4}, \frac{1}{6}$$
 a subsequence of $\{h_n\}_{n=4}^7$?

Is 8,6,2 a subsequence of g = 1,2,2,2,6,8,8,9?

Need to Identify a Sequence?

A great resource for sequences:

The Online Encyclopedia of Integer Sequences

(http://oeis.org/)

Example:

Let's try it! 2,3,5,7,11,13,17

Strings

Definition: <u>String</u>

A string is a contiguous finite sequence of 0 or more elements drawn from a set called the *alphabet*

Example:

A sequence of DNA nucleotides (e.g. ATTGACCT) is called a string.

A Java String also qualifies (alphabet: UNICODE values)

Strings

- Notation:
 - Lambda (λ) represents the empty (null) string
 - *xy* means strings *x* and *y* are concatenated
 - Superscripts denote repetition of concatenation
 - |x| represents the length of string x
 - A^* is the set of strings that can bw formed using elements of an alphabet A
 - A* is an infinite set
 - $\lambda \in A^*$

An observation about set cardinality:

Two sets A and B have the same cardinality **iff** there is a bijection from A to B

Definition: *<u>Finite</u>*

A set *S* is finite if there exists a bijective mapping between it and a set of cardinality |S|

Definition: <u>Countably Infinite (a.k.a. Denumerably Infinite)</u> A set is countably infinite if there exists a bijective

mapping between the set and either \mathbb{Z}^* or \mathbb{Z}^+

Definition: <u>Countable</u>

A set is countable if it is finite or countably infinite. (Otherwise, it is *uncountable*.)

Example:

Is the set of digits in the 'house number' of the Gould-Simpson building countable?

1040 E. 4th St.. $\Rightarrow \{0,1,4\}$

This set is finite, so, yes!

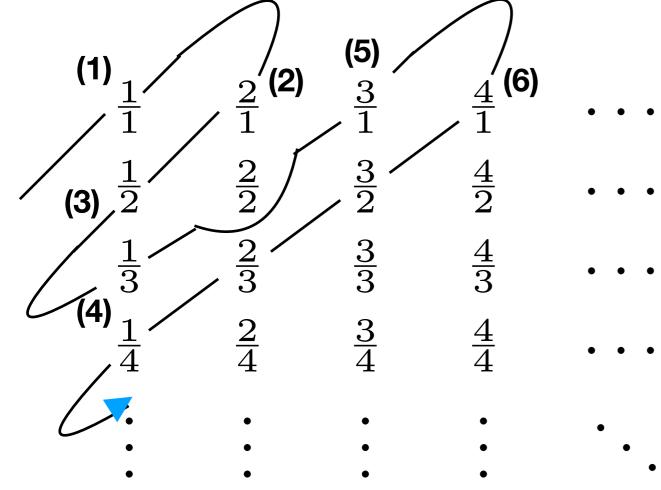
Is the set of positive multiples of 5 countable?

$$\mathbb{Z}^+:1$$
 2 3 4 ... $z = \frac{y}{5}$

5 10 15 20 ... 5*z* y

Invertable \rightarrow Bijection \rightarrow Countable! (So, yes!)

Question: Are the positive rational numbers countable?



Just skip duplicates $R = \{(1,\frac{1}{1}), (2,\frac{2}{1}), (3,\frac{1}{2}), (4,\frac{1}{3}), (5,\frac{3}{1}), ...\}$ $R^{-1} = \{(\frac{1}{1},1), (\frac{2}{1},2), (\frac{1}{2},3), (\frac{1}{3},4), (\frac{3}{1},5), ...\}$ It is invertable therefore

It is invertable, therefore it is a bijection.

Yes: (This is an application of a 'pairing function which invertible maps after duplicates are removed.) (With duplicates, the function is not invertable!)

(This is an example of a *boustrophedonic* path.)

<u>Conjecture</u>: A pairing function for \mathbb{R} cannot exist

Proof (Contradiction): Assume that a pairing function for the reals does exist. Given the set of real numbers, form a new number (not in the pairing) by changing the d^{th} digit of the d^{th} number. For example:

0.5491... won't be in pairing, because it has 1 digit different than every number in the pairing

The result must be a value not in the set of reals, yet it is a real. This is a contradiction (This is called *Cantor's Diagonal Argument*).

Therefore, a pairing function for $\mathbb R$ cannot exist